# 4. FREQUENCY RESPONSE FUNCTION DEVELOPMENT

## 4.1 Theory

All of the techniques discussed previously are useful if an analytical model of the system already exists. From an experimental point, this is rarely the case. Typically, solving problems on real systems or pieces of systems must be accomplished without the aid of a theoretical model or in order to verify a theoretical model.

In this chapter, frequency response function measurements will begin to be used as the basis for defining modal frequencies and damping values, modal vectors, modal mass, modal stiffness, and modal damping of real life structures. To accomplish this task, an analytical model will be developed to represent the transfer function between any possible measurement locations on the structure. Frequency response functions will be directly related to the transfer functions that have been theoretically developed.

The transfer function representation of an undamped multiple degree of freedom system can be formulated by starting with the differential equations of motion in terms of mass, stiffness, and damping matrices.

$$[\,M\,]\{\,\ddot{x}\,\} + [\,K\,]\{\,x\,\} = \{\,f\,\} \tag{4.1}$$

Taking the Laplace transform of Equation 4.1, assuming all initial conditions are zero, yields:

$$\left[\, s^2\,[\,M\,] + [\,K\,]\,\right]\{\,X(s)\,\} = \{\,F(s)\,\} \tag{4.2}$$

Let:

$$[\,B(s)\,] = \left[\, s^2\,[\,M\,] + [\,K\,]\,\right]$$

Then Equation 4.2 becomes:

$$[\,B(s)\,]\{\,X(s)\,\} = \{\,F(s)\,\} \tag{4.3}$$

where [ $B(s)$ ] is referred to as the *system impedance matrix* or just the *system matrix.*

Pre-multiplying Equation 4.3 by [ $B(s)$ ]$^{-1}$ yields:

$$[\ B(s)\ ]^{-1} \{\ F(s)\ \} = \{\ X(s)\ \}$$

Defining:

$$[\ H(s)\ ] = [\ B(s)\ ]^{-1}$$

Then:

$$[\ H(s)\ ] \{\ F(s)\ \} = \{\ X(s)\ \} \tag{4.4}$$

Equation 4.4 relates the system response *{ $X(s)$ }* to the system forcing functions *{ $F(s)$ }* through the matrix [ $H(s)$ ]. The matrix [ $H(s)$ ] is generally referred to as the *transfer function matrix.*

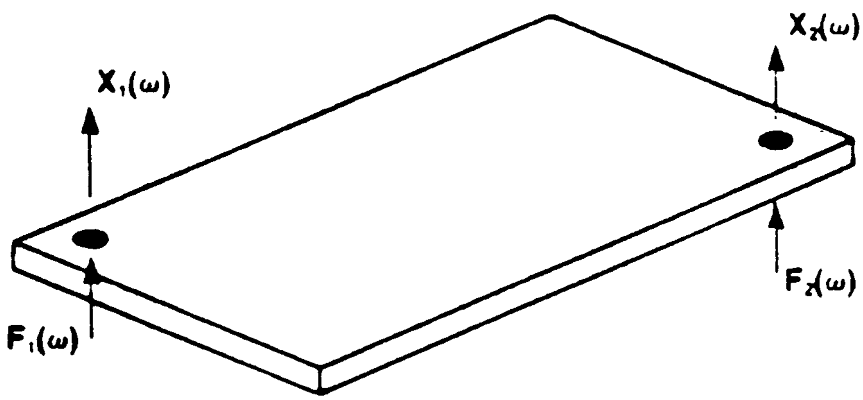


**Figure 4-1.** Two Input-Output Model

Equations 4.2-4.4 can be expanded for a two degree of freedom system in order to view in detail the components of each position in the system matrix.

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{4.5}$$

Taking the Laplace transform of Equation 4.5 yields:

$$\left[ \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} s^2 + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \right] \begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix}$$

Thus,

$$[\ B(s)\ ]\ \{\ X(s)\ \} = \{\ F(s)\ \}$$

where:

- $[\ B(s)\ ] = \begin{bmatrix} M_{11}\ s^2 + K_{11} & M_{12}\ s^2 + K_{12} \\ M_{21}\ s^2 + K_{21} & M_{22}\ s^2 + K_{22} \end{bmatrix}$

Defining the transfer function as the inverse of the impedance matrix:

$$[\ B(s)\ ]^{-1} = [\ H(s)\ ]$$

The inverse of the impedance matrix can be found for this analytical case as the adjoint of the impedance matrix divided by the determinant of the impedance matrix as follows:

$$[\ B(s)\ ]^{-1} = \frac{\begin{bmatrix} M_{22}\ s^2 + K_{22} & -(\ M_{12}\ s^2 + K_{12}\ ) \\ -(\ M_{21}\ s^2 + K_{21}\ ) & M_{11}\ s^2 + K_{11} \end{bmatrix}}{(M_{11}\ s^2 + K_{11})\ (M_{22}\ s^2 + K_{22}) - (M_{21}\ s^2 + K_{21})\ (M_{12}\ s^2 + K_{12})} \qquad (4.6)$$

Note that the denominator of Equation 4.6 is $|\ B(s)\ |$ which is the characteristic or frequency equation for the system. This highlights the fact that the complex-valued modal frequencies ($\lambda_r$) are global properties of the system since this characteristic equation appears in every term of $[\ H(s)\ ]$. This characteristic equation can be expressed as a product of its roots, thus:

$$|\ B(s)\ | = E\ (\ s - \lambda_1\ )\ (\ s - \lambda_2\ )\ (\ s - \lambda_3\ )\ (\ s - \lambda_4\ )$$

where:

- $E$ = Constant coeficient of the highest order term in the polynomial (Product/Sum of the mass terms)
- $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ are the 4 roots of the characteristic equation.

Equation 4.6 for a two degree of freedom system can then be written as:

$$\begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \frac{\begin{bmatrix} M_{22}\, s^2 + K_{22} & -(\, M_{12}\, s^2 + K_{12}\,) \\ -(\, M_{21}\, s^2 + K_{21}\,) & M_{11}\, s^2 + K_{11} \end{bmatrix}}{E\,(\, s - \lambda_1\,)\,(\, s - \lambda_2\,)\,(\, s - \lambda_3\,)\,(\, s - \lambda_4\,)} \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix} \tag{4.7}$$

$[\ H(s)\ ]$, the transfer function matrix, can be defined as:

$$[\ H\ (s)\ \ ] = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}$$

where, for instance:

$$H_{11}(s) = \frac{M_{22}\, s^2 + K_{22}}{E\,(\, s - \lambda_1\,)\,(\, s - \lambda_2\,)\,(\, s - \lambda_3\,)\,(\, s - \lambda_4\,)} \tag{4.8}$$

Equation 4.4 can now be expressed as:

$$\begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix} = \begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} \tag{4.9}$$

Multiplying out Equation 4.9 yields:

$$\begin{gathered} H_{11}(s)\ F_1(s) + H_{12}(s)\ F_2(s) = X_1(s) \\ H_{21}(s)\ F_1(s) + H_{22}(s)\ F_2(s) = X_2(s) \end{gathered} \tag{4.10}$$

If, in Equation 4.10, $F_2(s) = 0$, then:

$$H_{11}(s)\ F_1(s) = X_1(s)$$

$$H_{21}(s)\ F_1(s) = X_2(s)$$

This results in the familiar relationships for the transfer function (output over input):

$$H_{11}(s) = \frac{X_1(s)}{F_1(s)}$$

$$H_{21}(s) = \frac{X_2(s)}{F_1(s)}$$

In general

$$H_{pq} = \frac{X_p}{F_q}$$

where :

- $p$ is the output degree of freedom (physical location and orientation).
- $q$ is the input degree of freedom (physical location and orientation).

Therefore, measuring a column of the [ $H$ ] matrix is accomplished by using a single, fixed input (excitor system) with a roving response and measuring a row is accomplished by using a roving input (hammer) and a single fixed response. It should be reiterated that the subscript notation of $p$ or $q$ refers to both a physical location and also direction or orientation.

Thus, $H_{11}(s)$ is the transfer function measured by exciting the system with $F_1(s)$ and measuring the response $X_1(s)$. Similarly $H_{21}(s)$ is the transfer function measured by exciting the system with $F_1(s)$ and measuring the response at $X_2(s)$. Likewise, $H_{12}(s)$ and $H_{22}(s)$ can be measured by exciting the system with $F_2(s)$, letting $F_1(s) = 0$, and measuring the responses $X_1(s)$ and $X_2(s)$.

As in the single degree of freedom case, the denominator polynominal in Equation 4.8 is called the characteristic equation. Notice that all of the transfer functions that are represented in Equation 4.6 have the same denominator polynomial. The roots of this denominator polynomial (characteristic equation) are the modal frequencies of the system.

Since the coefficients of the characteristic equation are real, the roots will appear as complex conjugate pairs. Rewriting Equation 4.8 with this in mind gives the following result:

$$H_{11}(s) = \frac{M_{22}\ s^2 + K_{22}}{E\ (\ s - \lambda_1\ )\ (\ s - \lambda_1^*\ )\ (\ s - \lambda_2\ )\ (\ s - \lambda_2^*\ )} \tag{4.11}$$

where $\lambda_1$, $\lambda_1^*$, $\lambda_2$, and $\lambda_2^*$ are the *roots of the characteristic equation.* The roots of the characteristic equation are also referred to as the *poles of the transfer function* $H_{11}(s)$.

## 4.2 Analytical Model - Scalar/Matrix Polynomial (MDOF)

The general formulation of Equation 4.11 or of any of the transfer functions defined by an anlytical mass, damping and stiffness matrix model can be written as a numerator polynomial of the independent variable $s$ divided by a denominator polynomial of the independent variable $s$. Both polynomials involve coefficients that are different numerical combinations of the discrete values of mass, damping, stiffness of the system. The roots of the denominator polynomial are the modal frequencies of the system and are considered global properties of the system. The roots of the numerator polynomial are the zeroes of the system and are local properties of the system that depend upon the specific input-output relationship of the transfer function. The general polynomial model for a single transfer function can be written as follows using scalar coefficients:

$$\frac{X_p(s)}{F_q(s)} = H_{pq}(s) = \frac{\beta_n\ (s)^n + \beta_{n-1}\ (s)^{n-1} + \cdots + \beta_1\ (s)^1 + \beta_0\ (s)^0}{\alpha_m\ (s)^m + \alpha_{m-1}\ (s)^{m-1} + \cdots + \alpha_1\ (s)^1 + \alpha_0\ (s)^0} \tag{4.12}$$

The previous model can be rewritten in a more concise form as follows:

$$\frac{X_p(s)}{F_q(s)} = \frac{\sum_{k=0}^{n} \beta_k\ (s)^k}{\sum_{k=0}^{m} \alpha_k\ (s)^k}$$

Further rearrranging yields the following equation that is linear in the unknown $\alpha$ and $\beta$ terms:

$$\sum_{k=0}^{m} \alpha_k\ (s)^k X_p(s) = \sum_{k=0}^{n} \beta_k\ (s)^k F_q(s) \tag{4.13}$$

This model can be generalized to represent the general multiple input, multiple output case as follows using a matrix polynomial formulation:

$$\sum_{k=0}^{m} \left[ [\alpha_k] (s)^k \right] \{X(s)\} = \sum_{k=0}^{n} \left[ [\beta_k] (s)^k \right] \{F(s)\} \tag{4.14}$$

The previous models can be used to represent frequency response function (FRF) data by limiting the $s$ variable ($s = j\ \omega$) and applying a few matrix operations. If both sides of the above equation are post multiplied by the hermitian (complex conjugate transpose) of the force vector ($\{F(s)\}^H$), the following equation results.

$$\sum_{k=0}^{m} \left[ [\alpha_k] (s)^k \right] \{X(s)\} \{F(s)\}^H = \sum_{k=0}^{n} \left[ [\beta_k] (s)^k \right] \{F(s)\} \{F(s)\}^H$$

In the above equation note that the vector products are the definition of the cross power ($[G_{xf}(s)] = \{X(s)\}\{F(s)\}^H$) and auto power spectrum ($[G_{ff}(s)] = \{F(s)\}\{F(s)\}^H$) when $s = j\omega$ (after averaging).

$$\sum_{k=0}^{m} \left[ [\alpha_k] (s)^k \right] \left[ G_{xf}(s) \right] = \sum_{k=0}^{n} \left[ [\beta_k] (s)^k \right] \left[ G_{ff}(s) \right]$$

The above equation can be post multipled by the inverse of the auto power spectrum matrix.

$$\sum_{k=0}^{m} \left[ [\alpha_k] (s)^k \right] \left[ G_{xf}(s) \right] \left[ G_{ff}(s) \right]^{-1} = \sum_{k=0}^{n} \left[ [\beta_k] (s)^k \right] [I]$$

Finally, the above equation can be put in final form by noting that the product of the cross spectrum matrix and the inverse of the auto spectrum matrix ($[G_{xf}(s)]\ [G_{ff}(s)]^{-1}$) is the definition of the FRF matrix ($[H(s)]$) for the multiple input, multiple output case when $s = j\omega$.

$$\sum_{k=0}^{m} \left[ [\alpha_k] (j\omega)^k \right] [H(\omega)] = \sum_{k=0}^{n} \left[ [\beta_k] (j\omega)^k \right] [I] \tag{4.15}$$

For a single input, single output case the above equation yields:

$$\sum_{k=0}^{m} \alpha_k\, (j\omega)^k H_{pq}(\omega) = \sum_{k=0}^{n} \beta_k\, (j\omega)^k \tag{4.16}$$

## 4.3 Analytical Model - Partial Fraction (Residue)

Equation 4.11 can be represented very generally by expansion in terms of its partial fractions. That is:

$$H_{11}(s) = \frac{X_1(s)}{F_1(s)} = \frac{c_1}{(\, s - \lambda_1 \,)} + \frac{c_2}{(\, s - \lambda_1^* \,)} + \frac{c_3}{(\, s - \lambda_2 \,)} + \frac{c_4}{(\, s - \lambda_2^* \,)} \tag{4.17}$$

The constants $c_1 \rightarrow c_4$ can be found in a similar fashion as in the single degree of freedom case. Equating Equation 4.11 and Equation 4.17 yields:

$$\frac{M_{22}\; s^2 + K_{22}}{E\;(s - \lambda_1)\,(s - \lambda_1^*)\,(s - \lambda_2)\,(s - \lambda_2^*)} = \frac{c_1}{(s - \lambda_1)} + \frac{c_2}{(s - \lambda_1^*)} + \frac{c_3}{(s - \lambda_2)} + \frac{c_4}{(s - \lambda_2^*)} \tag{4.18}$$

Note that $c_1$ can be evaluated by multiplying Equation 4.18 by $s - \lambda_1$ and evaluating the expression at $s = \lambda_1$. Thus:

$$c_1 = \frac{M_{22}\; \lambda_1^2 + K_{22}}{E\;(\, \lambda_1 - \lambda_1^* \,)\,(\, \lambda_1 - \lambda_2 \,)\,(\, \lambda_1 - \lambda_2^* \,)} = A_{111} \tag{4.19}$$

In a similar fashion:

$$c_2 = \frac{M_{22}\; \lambda_1^{*2} + K_{22}}{E\;(\, \lambda_1^* - \lambda_1 \,)\,(\, \lambda_1^* - \lambda_2 \,)\,(\, \lambda_1^* - \lambda_2^* \,)} = c_1^* = A_{111}^*$$

$$c_3 = \frac{M_{22}\; \lambda_2^2 + K_{22}}{E\;(\, \lambda_2 - \lambda_1 \,)\,(\, \lambda_2 - \lambda_1^* \,)\,(\, \lambda_2 - \lambda_2^* \,)} = A_{112}$$

$$c_4 = \frac{M_{22}\; \lambda_2^{*2} + K_{22}}{E\;(\, \lambda_2^* - \lambda_1 \,)\,(\, \lambda_2^* - \lambda_1^* \,)\,(\, \lambda_2^* - \lambda_2 \,)} = c_3^* = A_{112}^*$$

Equation 4.17 becomes:

$$H_{11}(s) = \frac{A_{111}}{(\, s - \lambda_1 \,)} + \frac{A_{111}^*}{(\, s - \lambda_1^* \,)} + \frac{A_{112}}{(\, s - \lambda_2 \,)} + \frac{A_{112}^*}{(\, s - \lambda_2^* \,)} \tag{4.20}$$

Thus, the transfer function of a two degree of freedom system has been represented by the sum of two single degree of freedom systems (Section 2.5). This result can now be extrapolated to apply to any number of degrees of freedom.

Rewriting Equation 4.20 in terms of this summation:

$$H_{11}(s) = \sum_{r=1}^{2} \left[ \frac{A_{11r}}{s - \lambda_r} + \frac{A^*_{11r}}{s - \lambda^*_r} \right] \tag{4.21}$$

As in a single degree of freedom case, the $A_{11r}$'s are again referred to as the residues associated with the poles $\lambda_r$.

The rest of the transfer functions of the system can be expressed, using the same logic as the development of Equation 4.21.

$$H_{21}(s) = \sum_{r=1}^{2} \left[ \frac{A_{21r}}{s - \lambda_r} + \frac{A^*_{21r}}{s - \lambda^*_r} \right] \tag{4.22}$$

$$H_{12}(s) = \sum_{r=1}^{2} \left[ \frac{A_{12r}}{s - \lambda_r} + \frac{A^*_{12r}}{s - \lambda^*_r} \right] \tag{4.23}$$

$$H_{22}(s) = \sum_{r=1}^{2} \left[ \frac{A_{22r}}{s - \lambda_r} + \frac{A^*_{22r}}{s - \lambda^*_r} \right] \tag{4.24}$$

Equation 4.9 can now be rewritten in terms of the above partial fraction expansion.

$$\begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \begin{bmatrix} \sum_{r=1}^{2} \left[ \frac{A_{11r}}{s - \lambda_r} + \frac{A^*_{11r}}{s - \lambda^*_r} \right] & \sum_{r=1}^{2} \left[ \frac{A_{12r}}{s - \lambda_r} + \frac{A^*_{12r}}{s - \lambda^*_r} \right] \\ \sum_{r=1}^{2} \left[ \frac{A_{21r}}{s - \lambda_r} + \frac{A^*_{21r}}{s - \lambda^*_r} \right] & \sum_{r=1}^{2} \left[ \frac{A_{22r}}{s - \lambda_r} + \frac{A^*_{22r}}{s - \lambda^*_r} \right] \end{bmatrix} \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix} \tag{4.25}$$

The transfer function matrix [ $H(s)$ ] can also be rewritten in terms of partial fractions combining Equations 4.21 through 4.24.

$$[\ H(s)\ ] = \frac{\begin{bmatrix} A_{111} & A_{121} \\ A_{211} & A_{221} \end{bmatrix}}{(\ s - \lambda_1\ )} + \frac{\begin{bmatrix} A^*_{111} & A^*_{121} \\ A^*_{211} & A^*_{221} \end{bmatrix}}{(\ s - \lambda^*_1\ )} + \frac{\begin{bmatrix} A_{112} & A_{122} \\ A_{212} & A_{222} \end{bmatrix}}{(\ s - \lambda_2\ )} + \frac{\begin{bmatrix} A^*_{112} & A^*_{122} \\ A^*_{212} & A^*_{222} \end{bmatrix}}{(\ s - \lambda^*_2\ )} \tag{4.26}$$

The numerator matrices in each term of the above equation are called the residue matrices. Note that there is a separate matrix associated with each of the modal frequencies (poles) of the system. Note also that the residue matrices associated with complex conjugate modal frequencies are also complex conjugates.

Previously, the constant $A_{111}$ has been evaluated (Equation 4.19). The constants $A_{211}$, $A_{121}$, and $A_{221}$ can now be evaluated in a similar fashion. Once this is done, the form of the first residue matrix can be evaluated for pole $\lambda_1$.

$$A_{211} = \frac{-(\ M_{21}\ \lambda_1^2 + K_{21}\ )}{E\ (\ \lambda_1 - \lambda_1^*\ )\ i(\ \lambda_1 - \lambda_2\ )\ (\lambda_1 - \lambda_2^*\ )}$$

$$A_{121} = \frac{-(\ M_{12}\ \lambda_1^2 + K_{12}\ )}{E\ (\ \lambda_1 - \lambda_1^*\ )\ (\ \lambda_1 - \lambda_2\ )\ (\lambda_1 - \lambda_2^*\ )}$$

$$A_{221} = \frac{-(\ M_{11}\ \lambda_1^2 + K_{11}\ )}{E\ (\ \lambda_1 - \lambda_1^*\ )\ (\ \lambda_1 - \lambda_2\ )\ (\lambda_1 - \lambda_2^*\ )}$$

The first term on the right of Equation 4.27 can be rewritten as follows:

$$\frac{\begin{bmatrix} A_{111} & A_{121} \\ A_{211} & A_{221} \end{bmatrix}}{(\ s - \lambda_1\ )} = \frac{\begin{bmatrix} M_{22}\ \lambda_1^2 + K_{22} & -(\ M_{12}\ \lambda_1^2 + K_{12}\ ) \\ -(\ M_{21}\ \lambda_1^2 + K_{21}\ ) & M_{11}\ \lambda_1^2 + K_{11} \end{bmatrix}}{E\ (\ \lambda_1 - \lambda_1^*\ )\ (\ \lambda_1 - \lambda_2\ )\ (\ \lambda_1 - \lambda_2^*\ )\ (\ s - \lambda_1\ )} \tag{4.27}$$

Note that the numerator matrix on the right side of Equation 4.27 is the adjoint of the system matrix discussed in Section 3.8. Therefore, from Equation 3.41:

$$\begin{bmatrix} M_{22}\ \lambda_1^2 + K_{22} & -(\ M_{12}\ \lambda_1^2 + K_{12}\ ) \\ -(\ M_{21}\ \lambda_1^2 + K_{21}\ ) & M_{11}\ \lambda_1^2 + K_{11} \end{bmatrix} = \gamma_1 \begin{bmatrix} \psi_1\ \psi_1 & \psi_1\ \psi_2 \\ \psi_2\ \psi_1 & \psi_2\ \psi_2 \end{bmatrix}_1 \tag{4.28}$$

Plugging Equation 4.28 into Equation 4.27 gives:

$$\frac{\begin{bmatrix} A_{11\,1} & A_{12\,1} \\ A_{21\,1} & A_{22\,1} \end{bmatrix}}{(\, s - \lambda_1 \,)} = \frac{\gamma_1 \begin{bmatrix} \psi_1\,\psi_1 & \psi_1\,\psi_2 \\ \psi_2\,\psi_1 & \psi_2\,\psi_2 \end{bmatrix}_1}{E\,(\,\lambda_1 - \lambda_1^*\,)\,(\,\lambda_1 - \lambda_2\,)\,(\,\lambda_1 - \lambda_2^*\,)\,(\,s - \lambda_1\,)} \tag{4.29}$$

Finally, the relationship between the transfer function data (via the residue matrices) and the modal vectors of the system can be established. Equation 4.29 shows that the residue matrix has the same structure as the adjoint matrix: each column contains a redundant estimate of the same modal vector, different only by a constant. This relationship is normally stated as follows:

For pole $\lambda_1$ :

$$\begin{bmatrix} A_{11\,1} & A_{12\,1} \\ A_{21\,1} & A_{22\,1} \end{bmatrix} = Q_1 \begin{bmatrix} \psi_1\,\psi_1 & \psi_1\,\psi_2 \\ \psi_2\,\psi_1 & \psi_2\,\psi_2 \end{bmatrix}_1$$

In general, for pole $\lambda_r$ :

$$\begin{bmatrix} A_{11\,r} & A_{12\,r} \\ A_{21\,r} & A_{22\,r} \end{bmatrix} = Q_r \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_r \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_r^T = Q_r \begin{bmatrix} \psi_1\,\psi_1 & \psi_1\,\psi_2 \\ \psi_2\,\psi_1 & \psi_2\,\psi_2 \end{bmatrix}_r \tag{4.30}$$

where:

- $Q_1$ is a constant that is a function of the modal vector scaling and the absolute units of the residue matrix.
- $\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 =$ mode shape for pole $\lambda_1$

Equation 4.30 indicates that, with respect to the residue matrix, every row and column of the residue matrix contains the same modal vector multiplied by a component of the modal vector and the scaling constant. This is an important result for the experimental case. Therefore, assuming that all of the residues within a row or column are not perfectly zero, the modal vector can be estimated from only one row or column of the residue matrix. This will completely define the modal vector for that particular pole. In terms of the experimental requirements, in order to estimate a particular element in the residue matrix, a frequency response function must be measured. Under the assumption that every element in a particular row or column is not zero, only $N$ measurements will be required rather than $N \times N$. Once this information is measured, all other terms in the residue matrix could be synthesized.

This previous discussion represents the MINIMUM requirements in terms of measurements. In order to be certain that modal vectors are not missed or to utilize the redundant information in different rows or columns, more measurements than one row or column are typically taken.

It should be pointed out that the constant of proportionality $Q_r$ in Equation 4.30 is not unique since the constant will depend upon the choice of how the modal vector is scaled. This is consistent with the concept that modal vectors represent relative motion between the degrees of freedom. Note however that the residues are scaled quantities that depend upon the units of the frequency response function(s).

## 4.4 Modal Vector Example

The transfer function matrix for the two degree of freedom system example used previously can now be used to determine the modal vectors.

Substituting mass and stiffness values into Equation 4.6 gives:

$$[\ H(s)\ ] = \frac{\begin{bmatrix} (10\, s^2 + 6) & 2 \\ 2 & (5\, s^2 + 4) \end{bmatrix}}{(5\, s^2 + 4)\,(10\, s^2 + 6) - (-2)\,(-2)}$$

$$[\ H(s)\ ] = \frac{\begin{bmatrix} (10\, s^2 + 6) & 2 \\ 2 & (5\, s^2 + 4) \end{bmatrix}}{50\,(s^4 + 7/5\, s^2 + 2/5)}$$

The roots of the characteristic equation have previously been calculated as:

$$\lambda_1 = j\,\sqrt{2/5}\ \ (rad/sec) \qquad \lambda_1^* = -\,j\,\sqrt{2/5}\ \ (rad/sec)$$

$$\lambda_2 = j \ \ ( rad/sec ) \qquad \qquad \lambda_2^* = - j \ \ ( rad/sec )$$

Therefore:

$$[ \ H \ (s) \ \ ] = \frac{\begin{bmatrix} ( 10 \ s^2 + 6 ) & 2 \\ 2 & ( 5 \ s^2 + 4 ) \end{bmatrix}}{50 \ ( \ s - \ j \ \sqrt{2/5} \ ) \ ( \ s + \ j \ \sqrt{2/5} \ ) \ ( \ s - \ j \ ) \ ( \ s + \ j \ )}$$

The transfer function $H_{11}(s)$ can now be represented in terms of its partial fraction expansion:

$$H_{11}(s) = \frac{10 \ s^2 + 6}{50 \ ( \ s - \ j \ \sqrt{2/5} \ ) \ ( \ s + \ j \ \sqrt{2/5} \ ) \ ( \ s - \ j \ ) \ ( \ s + \ j \ )}$$

$$H_{11}(s) = \frac{A_{111}}{( \ s - \ j \ \sqrt{2/5} \ )} + \frac{A_{111}^*}{( \ s + \ j \ \sqrt{2/5} \ )} + \frac{A_{112}}{( \ s - \ j \ )} + \frac{A_{112}^*}{( \ s - \ j \ )}$$

$$A_{111} = - \ \frac{j \ \ \sqrt{2/5}}{12} \qquad \qquad A_{111}^* = \frac{j \ \ \sqrt{2/5}}{12}$$

$$A_{112} = \frac{- \ j}{15} \qquad \qquad A_{112}^* = \frac{j}{15}$$

In a similar fashion, the rest of the residues for the remaining transfer functions can also be determined.

The system transfer function matrix [ $H(s)$ ] can now be expressed in terms of partial fractions.

$$[\ H\ (s)\ \ ] = \frac{\begin{bmatrix} -\dfrac{j\sqrt{2/5}}{12} & -\dfrac{j\sqrt{2/5}}{12} \\ -\dfrac{j\sqrt{2/5}}{12} & -\dfrac{j\sqrt{2/5}}{12} \end{bmatrix}}{(\ s - j\sqrt{2/5}\ )} + \frac{\begin{bmatrix} \dfrac{j\sqrt{2/5}}{12} & \dfrac{j\sqrt{2/5}}{12} \\ \dfrac{j\sqrt{2/5}}{12} & \dfrac{j\sqrt{2/5}}{12} \end{bmatrix}}{(\ s + j\sqrt{2/5}\ )}$$

$$+ \frac{\begin{bmatrix} \dfrac{-j}{15} & \dfrac{j}{30} \\ \dfrac{j}{30} & \dfrac{-j}{60} \end{bmatrix}}{(\ s - j\ )} + \frac{\begin{bmatrix} \dfrac{j}{15} & \dfrac{-j}{30} \\ \dfrac{-j}{30} & \dfrac{j}{60} \end{bmatrix}}{(\ s + j\ )}$$

Recall that the modal vector associated with the pole frequency $\lambda_1$ is proportional to the residue matrix for pole $\lambda_1$. Notice that the first two residue matrices are just the complex conjugate of each other. Therefore, the modal vector for pole $\lambda_1^*$ is just the complex conjugate of the modal vector for pole $\lambda_1$. The same is true for the last two residue matrices. This will always be the case for conjugate pairs of poles.

The modal vector for the pole $\lambda_1 = \sqrt{2/5}$ can be extracted from the first residue matrix.

Using Equations 4.29 and 4.30, the modal vectors can be related to the residue matrix for the first pole.

$$Q_1\ \{\ \psi\ \ \}_1\ \{\ \psi\ \ \}_1^T = Q_1 \begin{bmatrix} \psi_1\ \psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{bmatrix}_1$$

$$Q_1\ \{\ \psi\ \ \}_1\ \{\ \psi\ \ \}_1^T = \begin{bmatrix} -\dfrac{j\sqrt{2/5}}{12} & -\dfrac{j\sqrt{2/5}}{12} \\ -\dfrac{j\sqrt{2/5}}{12} & -\dfrac{j\sqrt{2/5}}{12} \end{bmatrix}_1$$

The constant $Q_1 = \pm\ \dfrac{j\sqrt{2/5}}{12}$ can now be factored out of the residue matrix. The choice of this constant at this point is purely arbitrary by virtue of the fact that the absolute amplitudes of the modal vectors are purely arbitrary.

$$\begin{bmatrix} \psi_1\psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{bmatrix}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_1$$

Hence, the modal vector for the first mode is:

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}_1$$

Similarly, the modal vector for the second mode:

$$Q_2 \{ \psi \}_2 \{ \psi \}_2^T = Q_2 \begin{bmatrix} \psi_1 \psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{bmatrix}_2 = \begin{bmatrix} \frac{-j}{15} & \frac{j}{30} \\ \frac{j}{30} & \frac{-j}{60} \end{bmatrix}_2$$

The constant $Q_2 = \pm \frac{-j}{60}$ can now be factored from this residue matrix in the same arbitrary manner as before.

$$\begin{bmatrix} \psi_1 \psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{bmatrix}_2 = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}_2$$

Equating elements of the two matrices yields:

$$\psi_1\psi_1 = \psi_1^2 = 4 \qquad \psi_1 = 2$$

$$\psi_2\psi_1 = -2 \qquad \psi_2 = -1$$

Thus:

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_2 = \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}_2$$

Both of these modal vectors could have been obtained directly by just using either a row or a column of the residue matrix as the modal vector directly.

## 4.5 Analytical Model - General Partial Fraction (Residue)

Recalling Equation 4.21 for a two degree of freedom system:

$$H_{11}(s) = \sum_{r=1}^{2} \frac{A_{11r}}{(\ s - \lambda_r\ )} + \frac{A^*_{11r}}{(\ s - \lambda^*_r\ )}$$

Equation 4.21 can now be generalized for an $N$-degree of freedom system as the following:

$$H_{11}(s) = \sum_{r=1}^{N} \frac{A_{11r}}{(\ s - \lambda_r\ )} + \frac{A^*_{11r}}{(\ s - \lambda^*_r\ )} \qquad (4.31)$$

$$\begin{bmatrix}\ H_{pq}(s)\ \end{bmatrix} = \sum_{r=1}^{N} \frac{\begin{bmatrix}\ A_{pqr}\ \end{bmatrix}}{(\ s - \lambda_r\ )} + \frac{\begin{bmatrix}\ A^*_{pqr}\ \end{bmatrix}}{(\ s - \lambda^*_r\ )} \qquad (4.32)$$

Furthermore, the entire system transfer function matrix $[\ H(s)\ ]$ can be generalized as:

$$[\ H\ (s)\ \ ] = \sum_{r=1}^{N} \frac{[\ A\ ]_r}{s - \lambda_r} + \frac{[\ A^*\ ]_r}{s - \lambda^*_r} \qquad (4.33)$$

In terms of the modal vectors of the system directly:

$$[\ H(s)\ \ ] = \sum_{r=1}^{N} \frac{Q_r\ \{\ \psi\ \}_r\ \{\ \psi\ \}^T_r}{(\ s - \lambda_r\ )} + \frac{Q^*_r\ \{\ \psi\ \}^*_r\ \{\ \psi\ \}^{*T}_r}{(\ s - \lambda^*_r\ )} \qquad (4.34)$$

Equation 4.34 is the general form of the system transfer function matrix. As will be shown later, this form does not change when a system with damping is considered. Remember, though, that the complete transfer function can not be measured. Not once again that the frequency response function measurement, which is just Equation 4.34 evaluated at $s = j\ \omega$, is actually what is measured.

## 4.6 Residue Relationship to Modal Vectors

The relationship established in Section 4.3 between the residue matrix and the modal vector can be developed in a more formal or rigorous manner. It is important to understand that the residue defined in the previous section is the key to the relationship between modal vectors and modal scaling (modal mass) for the experimental case. At the present time, the following discussion is limited to the undamped and/or proportionally damped cases. The relationship for the case of general damping is discussed in a later section.

The development of the relationship between the residue matrix and the modal vectors proceeds along a similar path as the development of the relationship between the residue matrix and the adjoint of the system matrix (Section 3.8). Beginning with the definition of the impedance matrix:

$$[B(s)] = \Big[ [\ M\ ] s^2 + [\ C\ ] s + [\ K\ ] \Big] \tag{4.35}$$

where:

- $[\ B(s)\ ]$ = System Impedance Matrix

From matrix algebra:

$$[B(s)]\ [B(s)]^{-1} = [I] \tag{4.36}$$

Noting that the inverse of the impedance matrix is the transfer function matrix gives:

$$[B(s)]\ [H(s)] = [I] \tag{4.37}$$

Replacing the transfer function matrix with the equivalent partial fraction representation from Equation 4.33 yields:

$$[I] = \sum_{r=1}^{N} \frac{[\ B(s)\ ]\ [\ A\ ]_r}{s - \lambda_r} + \frac{[B(s)]\ [\ A^*\ ]_r}{s - \lambda_r^*} \tag{4.38}$$

Premultiplying each term of the above equation by $(s - \lambda_r)$ and evaluating the equation at $s = \lambda_r$

for any specific mode $r$ (all of the terms drop out except the term associated with $\lambda_r$):

$$[\ 0\ ] = \left[ B(\lambda_r) \right] [\ A\ ]_r \tag{4.39}$$

Note that the above equation proves that each column of the residue matrix $[A]_r$ must be proportional to the modal vector associated with $\lambda_r$ just as it did in the adjoint matrix case (Section 3.8). Likewise the structure of the residue matrix must be the same as the structure of the adjoint matrix.

$$[A]_r = Q_r \{\ \psi\ \}_r\ \{\ \psi\ \}_r^T \tag{4.40}$$

$$[A]_r = Q_r \begin{bmatrix} \psi_1\ \psi_1 & \psi_1\ \psi_2 & . & . & . & . & \psi_1\ \psi_N \\ \psi_2\ \psi_1 & \psi_2\ \psi_2 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \psi_N\ \psi_1 & \psi_N\ \psi_2 & . & . & . & . & \psi_N\ \psi_N \end{bmatrix}_r$$

where:

- $Q_r =$ constant associated with the scaling of $\{\psi\}_r$ relative to the absolute scaling (units) of the residue matrix.
- $Q_r =$ is proportional, but not generally equal to, the proportionality constant $\gamma_r$ defined in Section 3.8.

## 4.7 Residue Relationship to Modal Mass

For the proportionally damped case, which includes the undamped case as a trivial form, the relationship between the residue and the modal mass can also be established consistent with the modal mass found analytically from the mass matrix. Starting with Equation 4.38 and 4.40:

$$[I] = \sum_{r=1}^{N} \frac{[\ B(s)\ ]\ Q_r \{\psi\}_r\ \{\psi\}_r^T}{s - \lambda_r} + \frac{[B(s)]\ Q_r^* \left\{ \psi^* \right\}_r \left\{ \psi^* \right\}_r^T}{s - \lambda_r^*} \tag{4.41}$$

Note that, for proportionally damped systems, the modal vectors are always real (normal) modes. Therefore, the conjugate of a modal vector is the same as the modal vector ($\{\psi\}_r = \left\{\psi^*\right\}_r$). Making this substitution and premultiplying both sides of Equation 4.41 by $\{\psi\}_t^T$:

$$\{\psi\}_t^T = \sum_{r=1}^{N} \frac{\{\psi\}_t^T \; [\, B(s) \,] \, Q_r \{\psi\}_r \; \{\psi\}_r^T}{s - \lambda_r} + \frac{\{\psi\}_t^T \; [B(s)] \, Q_r^* \{\psi\}_r \; \{\psi\}_r^T}{s - \lambda_r^*} \tag{4.42}$$

Substitute Equation 4.35 into Equation 4.42:

$$\{\psi\}_t^T = \sum_{r=1}^{N} \frac{Q_r \; \{\psi\}_t^T \left[ \, [\, M \,] \, s^2 + [\, C \,] \, s + [\, K \,] \, \right] \{\psi\}_r \; \{\psi\}_r^T}{s - \lambda_r} +$$

$$\frac{Q_r^* \; \{\psi\}_t^T \left[ \, [\, M \,] \, s^2 + [\, C \,] \, s + [\, K \,] \, \right] \{\psi\}_r \; \{\psi\}_r^T}{s - \lambda_r^*} \tag{4.43}$$

Applying the orthogonality relationships between the modal vectors and the mass, damping and stiffness matrices eliminates all terms except for those associated with mode $t$:

$$\{\psi\}_t^T = \frac{Q_t \; (\; M_t \; s^2 + C_t \; s + K_t \;) \{\psi\}_t^T}{s - \lambda_t} + \frac{Q_t^* \; (\; M_t \; s^2 + C_t \; s + K_t \;) \{\psi\}_t^T}{s - \lambda_t^*} \tag{4.44}$$

Eliminating $\{\psi\}_t^T$ from each term of Equation 4.44 leaves the following scalar equation:

$$1 = \frac{Q_t \; (\; M_t \; s^2 + C_t \; s + K_t \;)}{s - \lambda_t} + \frac{Q_t^* \; (\; M_t \; s^2 + C_t \; s + K_t \;)}{s - \lambda_t^*} \tag{4.45}$$

Clearing the fractions from the previous equation:

$$(s - \lambda_t) \; (s - \lambda_t^*) = Q_t \; (\; M_t \; s^2 + C_t \; s + K_t \;) \; (s - \lambda_t^*) + Q_t^* \; (\; M_t \; s^2 + C_t \; s + K_t \;)(s - \lambda_t)$$

$$. \tag{4.46}$$

Note that, by definition:

$$( M_t \; s^2 + C_t \; s + K_t \; ) = M_t \; (s - \lambda_t) \; (s - \lambda_t^*) \tag{4.47}$$

Substituting Equation 4.47 into Equation 4.46 and eliminating the common terms on both sides of the equation $(s - \lambda_t) \; (s - \lambda_t^*)$:

$$1 = Q_t \; M_t \; (s - \lambda_t^*) + Q_t^* \; M_t \; (s - \lambda_t) \tag{4.48}$$

Evaluating Equation 4.49 at $s = \lambda_t$ gives the final relationship:

$$1 = Q_t \; M_t \; (\lambda_t - \lambda_t^*) \tag{4.49}$$

$$1 = Q_t \; M_t \; (2 \; j \; \omega_t) \tag{4.50}$$

$$M_t = \frac{1}{2 \; j \; \omega_t \; Q_t} \tag{4.51}$$

Equation 4.51 represents the relationship between modal mass and the scaling involved between the residues and the modal vectors (Recall Equation 4.40). Therefore, once the residue information is found for mode $t$ and some convenient form of modal vector scaling is chosen for mode $t$, the scaling constant $Q_t$ can be determined. Equation 4.51 can then be used to determine modal mass for mode $t$ consistent with the modal vector scaling. This means that as long as the modal vector is chosen consistently, modal mass can be compared between different solution approaches (analytical versus experimental, for example).