# 5. GENERAL DAMPED SYSTEMS

## 5.1 Non-Proportionally Damped Systems

In reality, physical structures or systems are generally comprised of many substructures tied together in various fashions. These substructures can be made-up of a variety of materials, i.e. metals, plastics, and wood. Furthermore, these substructures may be connected to one another by rivets, bolts, screws, dampers, springs, weldments, friction, etc. Also, the spatial geometry of the structure may be very complicated, such as an exhaust system of an automobile. All of these factors influence the inherent dynamical properties of the structure. For these structures, mass, damping, and stiffness distribution (matrices) of the system are rather complicated. In general, for real life structures, the damping matrix for such a system will not always be proportional to the mass and/or stiffness matrix. Therefore, the damping of this system can be classified as non-proportional.

In an analytical sense, modal analysis of this general type of damped system cannot be described using the formulation of the eigenvalue problem as discussed previously for an undamped system. Remember, that in an analytical sense, the purpose of modal analysis is to find a coordinate transformation that uncoupled the original equations of motion. This coordinate transformation turned out to be a matrix comprised of the modal vectors of the system. These modal vectors were determined from the solution of the eigenvalue problem for that system. The coordinate transformation diagonalized the system mass, damping, and stiffness matrices, for an undamped or proportionally damped system. When a system contains non-proportional damping, the previously used formulation of the eigenvalue problem will not yield modal vectors (eigenvectors) that uncouple the equations of motion of the system. A technique used to circumvent this problem was first documented by Duncan and Collar and involves the reformulation of the original equations of motion, for an $N$-degree of freedom system, into an equivalent set of $2N$ first order differential equations known as Hamilton's Canonical Equations. The solution of these equations can be carried out in a similar manner that has been discussed previously.

For nonproportional damping, the coordinate transformation discussed previously, that diagonalizes the system mass and stiffness matrices, will not diagonalize the system damping matrix. Therefore, when a system with nonproportional damping exists, the equations of motion are coupled when formulated in $N$ dimension physical space. Fortunately, the equations of motion can be uncoupled when formulated in $2N$ dimension *state space.* This is accomplished by augmenting the original $N$ dimension physical space equation by a $N$ dimension identity as follows.

Assume a viscous, nonproportionally damped system can be represented by Equation 5.1.

$$[\,M\,]\{\,\ddot{x}\,\} + [\,C\,]\{\,\dot{x}\,\} + [\,K\,]\{\,x\,\} = \{\,f\,\} \tag{5.1}$$

This system of equations can be augmented by the identity shown in Equation 5.2:

$$[\,M\,]\{\,\dot{x}\,\} - [\,M\,]\{\,\dot{x}\,\} = \{\,0\,\} \tag{5.2}$$

Equations 5.1 and 5.2 can be combined as follows to yield a new system of $2N$ equations. Note that all the matrices in Equation 5.3 are symmetric and Equation 5.3 is now in a classical eigenvalue solution form. The notation used in Equation 5.3 is consistent with the notation used in many mathmatics and/or controls textbooks.

$$[\,A\,]\{\,\dot{y}\,\} + [\,B\,]\{\,y\,\} = \left\{\,f'\,\right\} \tag{5.3}$$

where:

- $[A\,] = \begin{bmatrix} [\,0\,] & [\,M\,] \\ [\,M\,] & [\,C\,] \end{bmatrix} \qquad [B\,] = \begin{bmatrix} -[\,M\,] & [\,0\,] \\ [\,0\,] & [\,K\,] \end{bmatrix}$

- $\{\,\dot{y}\,\} = \begin{Bmatrix} \{\ddot{x}\} \\ \{\dot{x}\} \end{Bmatrix} \qquad \{\,y\,\} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} \qquad \left\{\,f'\,\right\} = \begin{Bmatrix} \{0\} \\ \{\,f\,\} \end{Bmatrix}$

Forming the homogeneous equation from Equation 5.3 yields:

$$[\,A\,]\{\,\dot{y}\,\} + [\,B\,]\{\,y\,\} = \{\,0\,\} \tag{5.4}$$

The solution of Equation 5.4 yields the complex-valued natural frequencies (eigenvalues) and complex-valued modal vectors (eigenvectors) for the augmented $2N$ equation system. Note that in this mathematical form, the eigenvalues will be found directly (not the square of the eigenvalue) and the $2N$ eigenvectors will be $2N$ in length. The exact form of the eigenvectors can be seen from the eigenvector matrix (state space modal matrix) for the $2N$ equation system. Note that the notation $\{\ \phi\ \}$ is used for an eigenvector in the $2N$ equation system and that the notation $\{\ \psi\ \}$ is used for an eigenvector of the original $N$ equation system.

The state space modal matrix $[\ \phi\ ]$ for this nonproportionally damped system can now be assembled.

$$\begin{bmatrix} \phi \end{bmatrix} = \begin{bmatrix} \{\phi\}_1 & \{\phi\}_2 & \{\phi\}_3 & \{\cdots\} & \{\phi\}_r & \{\cdots\} & \{\phi\}_{2N} \end{bmatrix} \tag{5.5a}$$

Based upon the change of coordinate applied in Equation 5.3, each column of the eigenvector matrix (each eigenvector) is made up of the derivative of the desired modal vector above the desired modal vector. This structure is shown in Equation 5.5b.

$$\begin{bmatrix} \phi \end{bmatrix} = \begin{bmatrix} \lambda_1\{\psi\}_1 & \lambda_2\{\psi\}_2 & \lambda_3\{\psi\}_3 & \cdots & \lambda_r\{\psi\}_r & \cdots & \lambda_{2N}\{\psi\}_{2N} \\ \{\psi\}_1 & \{\psi\}_2 & \{\psi\}_3 & \cdots & \{\psi\}_r & \cdots & \{\psi\}_{2N} \end{bmatrix} \tag{5.5b}$$

### 5.1.1 Weighted Orthogonality of the Eigenvectors

Similar to the case for undamped systems, a set of weighted orthogonality relationships are valid for the system matrices $[A]$ and $[B]$.

$$\{\phi\}_r^T\ [A]\ \{\phi\}_s = 0 \tag{5.6}$$

$$\{\phi\}_r^T\ [B]\ \{\phi\}_s = 0 \tag{5.7}$$

The terms *modal A* and *modal B* can now be defined as follows. Note that these quantities have the same properties as *modal mass* and *modal stiffness* for the undamped and proportionally damped cases.

$$\{\phi\}_r^T\ [A]\ \{\phi\}_r = M_{A_r} \tag{5.8}$$

$$\{\phi\}_r^T\ [B]\ \{\phi\}_r = M_{B_r} \tag{5.9}$$

The terms *modal A* and *modal B* are modal scaling factors for the nonproportional case just as modal mass and modal stiffness can be used for the undamped and proportionally damped cases. Whenever complex modal vectors are present, *modal A* and *modal B* should be used to provide the modal scaling. Note that *modal A* and *modal B* could be used to provide modal scaling even for the undamped and proportionally damped cases.

Note that the eigenvector matrix (state space modal matrix) provides a coordinate transformation from the physical state space coordinate system to the uncoupled principl state space coordinate system.

$$[\,A\,]\{\,\dot{y}\,\} + [\,B\,]\{\,y\,\} = \left\{\,f'\,\right\} \qquad (5.10)$$

$$[\,A\,]\left[\,\phi\,\right]\{\,\dot{q}\,\} + [\,B\,]\left[\,\phi\,\right]\{\,q\,\} = \left\{\,f'\,\right\} \qquad (5.11)$$

$$\left[\,\phi\,\right]^T[\,A\,]\left[\,\phi\,\right]\{\,\dot{q}\,\} + \left[\,\phi\,\right]^T[\,B\,]\left[\,\phi\,\right]\{\,q\,\} = \left[\,\phi\,\right]^T\left\{\,f'\,\right\} \qquad (5.12)$$

For the $r-th$ eigenvalue/eigenvector:

$$M_{A_r}\,\dot{q}_r + M_{B_r} = f_r' \qquad (5.13)$$

This uncoupled equation has a characteristic equation of the form:

$$M_{A_r}\,s + M_{B_r} = 0 \qquad (5.14)$$

This means that Modal A and Modal B for each mode are related.

$$M_{A_r}\,\lambda_r + M_{B_r} = 0 \qquad (5.15)$$

$$M_{A_r}\,\lambda_r = -\,M_{B_r} \qquad (5.16)$$

This concept will not be pursued further here. The interested reader is referred to Chapter 6 of *Mechanical Vibrations* by Tse, Morse and Hinkle or Chapter 9 of *Analytical Methods in*

*Vibrations* by Leonard Meirovitch.

In an experimental sense, the approach to the problem is the same for non-proportionally damped system as for an undamped or a proportionally damped system. No matter what type of damping a structure has, proportional or non-proportional, the frequency response functions of the system can be measured. For this reason, the modal vectors of the system can be found by using the residues determined from the frequency response function measurements. While the approach is the same as before, the results in terms of modal vectors will be somewhat more complicated for a system with non-proportional damping. As an example of the differences that result for the non-proportional case, the same two degree of freedom system, used in previous examples, will be used again except that the damping matrix will be made non-proportional to the mass and/or stiffness matrices.

## 5.2 Proportionally Damped Systems

For the class of physical damping mechanisms that can be mathematically represented by the proportional damping concept, the coordinate transformation discussed previously for the undamped case, that diagonalizes the system mass and stiffness matrices, will also diagonalize the system damping matrix. Therefore, when a system with proportional damping exists, that system of coupled equations of motion can be transformed as before to a system of equations that represent an uncoupled system of single degree of freedom systems that are easily solved. The procedure to accomplish this follows.

Assume a viscously damped system can be represented by Equation 5.17.

$$[\,M\,]\{\,\ddot{x}\,\} + [\,C\,]\{\,\dot{x}\,\} + [\,K\,]\{\,x\,\} = \{\,f\,\} \tag{5.17}$$

The eigenvalue problem associated with the undamped system can be solved as a first step to understanding the problem.

$$[\,M\,]\{\,\ddot{x}\,\} + [\,K\,]\{\,x\,\} = \{\,0\,\}$$

This yields the system's natural frequencies (eigenvalues) and modal vectors (eigenvectors).

The modal matrix $[\psi]$ for this undamped system can now be assembled.

$$\left[\ \psi\ \right] = \left[\ \{\ \psi\ \}_1\ \{\ \psi\ \}_2\ \ldots\ \{\ \psi\ \}_N\ \right]$$

The coordinate transformation can now be applied to Equation 5.10:

$$[\ x\ ] = \left[\ \psi\ \right]\{\ q\ \}$$

$$[\ M\ ]\left[\ \psi\ \right]\{\ \ddot{q}\ \} + [\ C\ ]\left[\ \psi\ \right]\{\ \dot{q}\ \} + [\ K\ ]\left[\ \psi\ \right]\{\ q\ \} = \{\ f\ \}$$

Now pre-multiply Equation 5.18 by $[\psi]^T$.

$$\left[\ \psi\ \right]^T[\ M\ ]\left[\ \psi\ \right]\{\ \ddot{q}\ \} + \left[\ \psi\ \right]^T[\ C\ ]\left[\ \psi\ \right]\{\ \dot{q}\ \}$$

$$+ \left[\ \psi\ \right]^T[\ K\ ]\left[\ \psi\ \right]\{\ q\ \} = \left[\ \psi\ \right]^T\{\ f\ \} \qquad (5.19)$$

Due to the orthogonality properties of the modal vectors :

$$\left[\ \psi\ \right]^T[\ M\ ]\left[\ \psi\ \right] = \lceil M \rfloor$$

$$\left[\ \psi\ \right]^T[\ K\ ]\left[\ \psi\ \right] = \lceil K \rfloor$$

Since the assumed form of the damping matrix is proportional to the mass and/or stiffness matrix, the damping matrix will also be diagonalized.

$$[\ C\ ] = \alpha\,[\ M\ ] + \beta\,[\ K\ ]$$

The application of the orthogonality condition yields:

$$\left[\ \psi\ \right]^T[\ C\ ]\left[\ \psi\ \right] = \left[\ \psi\ \right]^T\{\ \alpha\,[\ M\ ] + \beta\,[\ K\ ]\ \}\left[\ \psi\ \right]$$

$$\left[ \psi \right]^T \left[ C \right] \left[ \psi \right] = \alpha \left[ \psi \right]^T \left[ M \right] \left[ \psi \right] + \beta \left[ \psi \right]^T \left[ K \right] \left[ \psi \right]$$

$$\left[ \psi \right]^T \left[ C \right] \left[ \psi \right] = \alpha \left\lceil M \right\rfloor + \beta \left\lceil K \right\rfloor$$

Therefore:

$$\left[ \psi \right]^T \left[ C \right] \left[ \psi \right] = \left\lceil C \right\rfloor$$

where:

- $\left\lceil C \right\rfloor$ is a diagonal matrix.

Therefore, Equation 5.19 becomes:

$$\left\lceil M \right\rfloor \{ \ddot{q} \} + \left\lceil C \right\rfloor \{ \dot{q} \} + \left\lceil K \right\rfloor \{ q \} = \left[ \psi \right]^T \{ f(t) \} \tag{5.20}$$

Equation 5.20 represents an uncoupled set of damped single degree of freedom systems. The $r$-th equation of Equation 5.20 is:

$$M_r \ddot{q}_r + C_r \dot{q}_r + K_r q_r = f_r^{\prime}(t) \tag{5.21}$$

Equation 5.21 is the equation of motion for a system represented below.

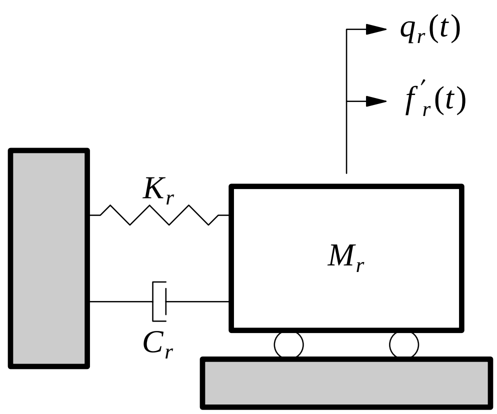


**Figure 5-1.** Proportional Damped SDOF Equivalent Model

The solution of this damped single degree of freedom system has been discussed previously.

## 5.3 Example with Proportional Damping

In order to understand the concept of proportional damping, the same two degree of freedom example, worked previously for the undamped case, can be reworked for the proportionally damped case.

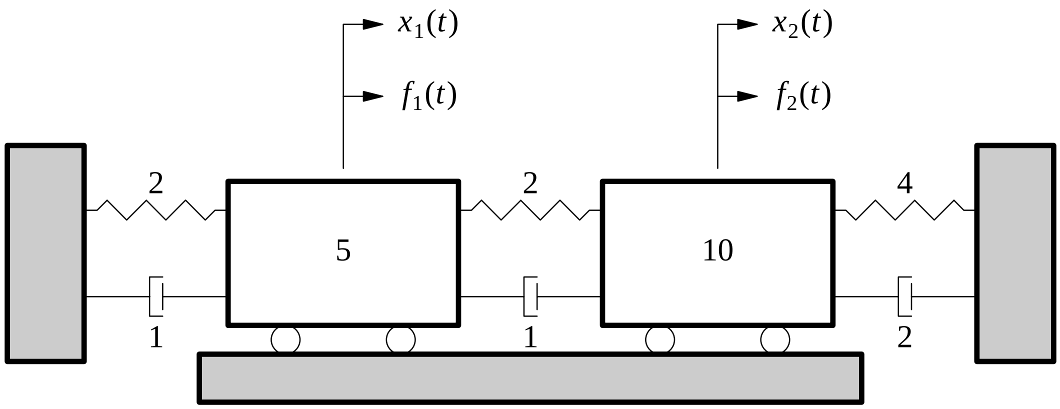


**Figure 5-2.** Two Degree of Freedom Model with Proportional Damping

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Note that the damping matrix $[\ C\ ]$ is proportional to the stiffness matrix $[\ K\ ]$:

$$[\ C\ ] = \left(\frac{1}{2}\right)[\ K\ ]$$

While this form of the damping matrix is quite simple, the solution that will result will yield a general characteristic that is common to all problems that can be described by the concept of proportional damping.

Using the previously calculated modal vectors and natural frequencies for the undamped system, a coordinate transformation can be performed:

$$\{\, x \,\} = \left[\, \psi \,\right] \{\, q \,\}$$

Noting the results of the previous example in Section 3.6, the uncoupling of the equations of motion for the mass and stiffness matrices has already been shown. Therefore, for the damping matrix:

$$\left[\, \psi \,\right]^T \left[\, C \,\right] \left[\, \psi \,\right] = \begin{bmatrix} \sqrt{1/15} & \sqrt{1/15} \\ \sqrt{2/15} & -\frac{1}{2}\sqrt{2/15} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{1/15} & \sqrt{2/15} \\ \sqrt{1/15} & -\frac{1}{2}\sqrt{2/15} \end{bmatrix}$$

$$\left[\, \psi \,\right]^T \left[\, C \,\right] \left[\, \psi \,\right] = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/2 \end{bmatrix}$$

The transformed system in terms of modal coordinates would be:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 1/5 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2/5 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} f_1'(t) \\ f_2'(t) \end{Bmatrix}$$

or pictorially:

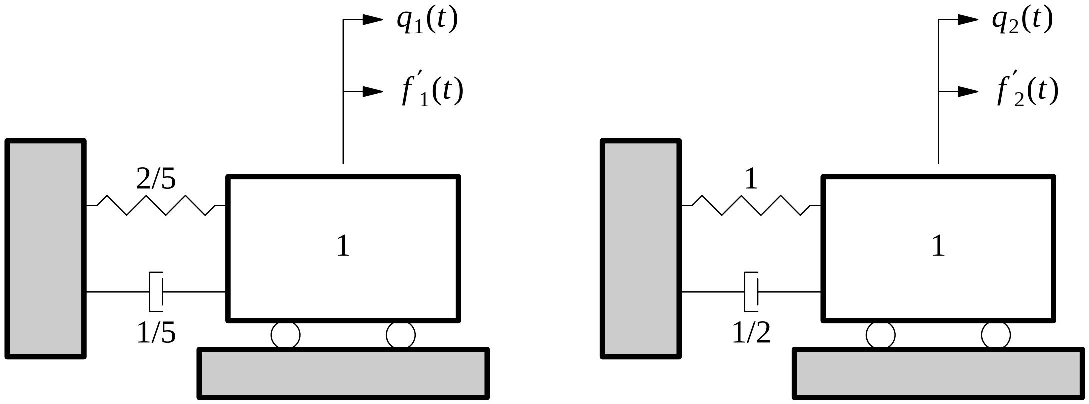


**Figure 5-3.** Proportionally Damped MDOF Equivalent Model

### 5.3.1 Frequency Response Function Implications

This same problem can be evaluated from a transfer function point of view by determining the modal frequencies and modal vectors. The transfer function matrix of the system is:

$$[\ H(s)\ ] = \frac{\begin{bmatrix} (M_{22}\, s^2 + C_{22}\, s + K_{22}) & -(M_{12}\, s^2 + C_{12}\, s + K_{12}) \\ -(M_{21}\, s^2 + C_{21}\, s + K_{21}) & (M_{11}\, s^2 + C_{11}\, s + K_{11}) \end{bmatrix}}{|\ B\ (s)\ |}$$

$$|\ B\ (s)\ | = \left(M_{11}\ s^2 + C_{11}\ s + K_{11}\right)\left(M_{22}\ s^2 + C_{22}\ s + K_{22}\right) - \left(M_{21}\ s^2 + C_{21}\ s + K_{21}\right)\left(M_{12}\ s^2 + C_{12}\ s + K_{12}\right)$$

Upon substituting the values for the mass, damping, and stiffness matrices from the example problem:

$$[\ H(s)\ ] = \frac{\begin{bmatrix} \left(10\, s^2 + 3\, s + 6\right) & (s + 2) \\ (s + 2) & \left(5\, s^2 + 2\, s + 4\right) \end{bmatrix}}{50\, s^4 + 35\, s^3 + 75\, s^2 + 20\, s + 20}$$

The roots of the characteristic equation are:

$$\lambda_1 = -\frac{1}{10} + j\,\frac{\sqrt{39}}{10}\ (\ rad/sec\ ) \qquad \lambda_1^* = -\frac{1}{10} - j\,\frac{\sqrt{39}}{10}\ (\ rad/sec\ )$$

$$\lambda_2 = -\frac{1}{4} + j\,\frac{\sqrt{15}}{4}\ (\ rad/sec\ ) \qquad \lambda_2^* = -\frac{1}{4} - j\,\frac{\sqrt{15}}{4}\ (\ rad/sec\ )$$

Note that the poles of a damped system now contain a real part. The pole of a transfer function has been previously defined (Chapter 2) as:

$$\lambda_r = \sigma_r + j\,\omega_r$$

where:

- $\sigma_r$ = damping factor
- $\omega_r$ = damped natural frequency.

Remember that in the previous undamped case, the poles could be written in the following form:

$$\lambda_r = j\,\omega_r$$

where:

- $\omega_r$ = damped natural frequency = undamped natural frequency

The system modal vectors can be determined by just using the minimum transfer function data, assuming that all elements of the row or column are not perfectly zero. If this is true, then the modal vectors can be found by using one row or column out of the system transfer function matrix. For example, choose $H_{11}\ (s)$ and $H_{21}\ (s)$, which amounts to the first column.

First, $H_{11}\ (s)$ and $H_{21}\ (s)$ can be expanded in terms of partial fractions.

$$H_{11}\ (s) = \frac{10\ s^2 + 3\ s + 6}{50\ (\ s - \lambda_1\ )\ (\ s - \lambda_1^*\ )\ (\ s - \lambda_2\ )\ (\ s - \lambda_2^*\ )}$$

After some work:

$$H_{11}\ (s) = \frac{-\ j\ \frac{\sqrt{39}}{117}}{(\ s - \lambda_1\ )} + \frac{j\ \frac{\sqrt{39}}{117}}{(\ s - \lambda_1^*\ )} + \frac{-\ j\ \frac{4\ \sqrt{15}}{225}}{(\ s - \lambda_2\ )} + \frac{j\ \frac{4\ \sqrt{15}}{225}}{(\ s - \lambda_2^*\ )}$$

$$H_{21}\ (s) = \frac{s + 2}{50\ (\ s - \lambda_1\ )\ (\ s - \lambda_1^*\ )\ (\ s - \lambda_2\ )\ (\ s - \lambda_2^*)}$$

$$H_{21}\ (s) = \frac{-\ j\ \frac{\sqrt{39}}{117}}{(\ s - \lambda_1\ )} + \frac{j\ \frac{\sqrt{39}}{117}}{(\ s - \lambda_1^*\ )} + \frac{j\ \frac{2\ \sqrt{15}}{225}}{(\ s - \lambda_2\ )} + \frac{-\ j\ \frac{2\ \sqrt{15}}{225}}{(\ s - \lambda_2^*\ )}$$

Recall that the residues have been shown to be proportional to the modal vectors. Since only a column of the transfer function has been used from the transfer function matrix, only a column of the residue matrix for each modal frequency has been calculated.

Thus, by looking at the residues for the first pole $\lambda_1$, the modal vector that results is:

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 \rightarrow \begin{Bmatrix} \frac{-j\ \sqrt{39}}{117} \\ \frac{-j\ \sqrt{39}}{117} \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}_1 \tag{5.23}$$

For the modal frequency $\lambda_2$, the modal vector that results is:

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_2 \rightarrow \begin{Bmatrix} \frac{-j\ 4\ \sqrt{15}}{225} \\ \frac{2\ j\ \sqrt{15}}{225} \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix}_2 \tag{5.24}$$

Thus, the modal vectors calculated from the transfer function matrix of this proportionally damped system are the same as for the undamped system. This will always be the case as long as the system under consideration exhibits proportional type of damping.

Notice that for the undamped and proportionally damped systems studied, not only were their modal vectors the same, but their modal vectors, when normalized, are real valued. This may not be the case for real systems. Since the modal vectors are real valued, they are typically referred to as *real modes or normal modes.*

## 5.3.2 Impulse Response Function Considerations

The impulse response function of a proportionally damped multiple degree of freedom system can now be discussed. Remember that the impulse response function is the time domain equivalent to the transfer function. Once the impulse response function for a multiple degree of freedom system has been formulated, it can be compared to the previous single degree of freedom system. Recall Equation 4.24.

$$H_{pq}(s) = \sum_{r=1}^{N} \frac{A_{pqr}}{(s - \lambda_r)} + \frac{A^*_{pqr}}{(s - \lambda^*_r)} \tag{5.25}$$

Recall the definition of $H_{pq}(s)$:

$$H_{pq}(s) = \frac{X_p(s)}{F_q(s)}$$

Therefore:

$$X_p(s) = F_q(s) \sum_{r=1}^{N} \frac{A_{pqr}}{(s - \lambda_r)} + \frac{A^*_{pqr}}{(s - \lambda^*_r)} \tag{5.26}$$

If the force used to excite the DOF ($q$) is an impulse, then the Laplace transform will be unity. While this concept is used to define the impulse response function, once the transfer function is formulated and the system is considered to be linear, this type of assumption can be made with complete generality.

$$F_q(s) = 1$$

Thus the system impulse response at DOF $(p)$ is the inverse Laplace transform of Equation 5.19.

$$h_{pq}(t) = \mathbf{L}^{-1}\left\{ X_p(s) \right\} = \mathbf{L}^{-1}\left\{ \sum_{r=1}^{N} \frac{A_{pqr}}{(s - \lambda_r)} + \frac{A^*_{pqr}}{(s - \lambda^*_r)} \right\}$$

If $p = q$, this would be the driving point impulse response function. For simplicity, Equation 5.27 can be expanded for the two degree of freedom case (N = 2).

$$X_p(s) = \frac{A_{pp1}}{(s - \lambda_1)} + \frac{A^*_{pp1}}{(s - \lambda^*_1)} + \frac{A_{pp2}}{(s - \lambda_2)} + \frac{A^*_{pp2}}{(s - \lambda^*_2)} \tag{5.28}$$

$$h_{pp}(t) = \mathbf{L}^{-1}\left\{ X_p(s) \right\} = A_{pp1}\, e^{\lambda_1 t} + A^*_{pp1}\, e^{\lambda^*_1 t} + A_{pp2}\, e^{\lambda_2 t} + A_{pp2}\, e^{\lambda^*_2 t} \tag{5.29}$$

Recall that:

$$\lambda_r = \sigma_r + j\,\omega_r$$

$$\lambda^*_r = \sigma_r - j\,\omega_r$$

Also, since the system is proportionally damped, the residues are all purely imaginary. Therefore, the following definition can be made arbitrarily. This definition will make certain trigonometric identities obvious in a later equation.

$$A_{pp1} = \frac{R_{pp1}}{2\,j}$$

$$A_{pp2} = \frac{R_{pp2}}{2\,j}$$

Using Euler's formula, Equation 5.22 then becomes:

$$h_{pp}\,(t) \;=\; \frac{R_{pp1}}{2\,j}\left(\, e^{(\,\sigma_1\,+\,j\,\omega_1\,)\,t} \;-\; e^{(\,\sigma_1\,-\,j\,\omega_1\,)\,t} \,\right)$$

$$+\;\frac{R_{pp2}}{2\,j}\left(\, e^{(\,\sigma_2\,+\,j\,\omega_2\,)\,t} \;-\; e^{(\,\sigma_2\,-\,j\,\omega_2\,)\,t} \,\right)$$

$$h_{pp}\,(t) \;=\; R_{pp1}\, e^{\sigma_1\,t}\, \frac{(\,e^{j\,\omega_1\,t} \;-\; e^{\,j\,-\omega_1\,t}\,)}{2\,j} \;+\; R_{pp2}\, e^{\sigma_2\,t}\, \frac{(\,e^{j\,\omega_2\,t} \;-\; e^{-\,j\,\omega_2\,t}\,)}{2\,j}$$

$$h_{pp}\,(t) \;=\; R_{pp1}\;\; e^{\sigma_1\,t}\;\; \sin\,(\,\omega_1\,t\,) \;+\; R_{pp2}\, e^{\sigma_2 t}\;\; \sin\,(\,\omega_2\,t\,)$$

Finally:

$$h_{pp}\,(t) \;=\; R_{pp1}\, e^{\sigma_1\,t}\;\sin(\,\omega_1\,t\,) \;+\; R_{pp2}\, e^{\sigma_2\,t}\;\sin(\,\omega_2\,t\,) \tag{5.30}$$

Comparing this to the single degree of freedom case done previously, note that the impulse response function is nothing more than the summation of two single degree of freedom responses. Note also that the amplitude of the impulse response function is directly related to the residues for the two modal vectors. This means that the impulse response function amplitude is directly related to the modal vectors of the two modes, since the residues have been shown to be directly proportional to the modal vectors.

In summary, a transfer function, and therefore a frequency response function, can be expressed as a sum of single degree of freedom systems (Equation 4.24). A typical frequency response function is illustrated in Figure (5-4), in terms of its real and imaginary parts. This frequency response function represents the response of the system at degree of freedom 1 due to a force applied to the system at degree of freedom 2 $H_{12}(\omega)$. Figure (5-5) is the equivalent impulse response function of Figure (5-4). Likewise, this impulse response function represents the response of the system at degree of freedom 1 due to a unit impulse applied to the system at degree of freedom 2 $h_{12}(t)$. Notice that the impulse response function starts out at zero for t = 0 as it should for a system with real modes of vibration. Since the transfer function is the sum of single degree of freedom systems, the frequency response functions for each of the single degree of freedom systems can be plotted independently as is illustrated in Figures (5-6) and (5-8).

Note that by adding together Figures (5-6) and (5-8), the same plot as in Figure (5-4) would result. In Figures (5-6) and (5-8) the imaginary parts peak where the real parts cross zero, where as, in Figure (5-4) this is not exactly true. Similarly, the impulse response functions for each single degree of freedom system can be plotted separately as in Figures (5-7) and (5-9). The sum of these two figures would be the same as Figure (5-5). Therefore, the impulse response of a two degree of freedom system is just the sum of two damped sinusoids.

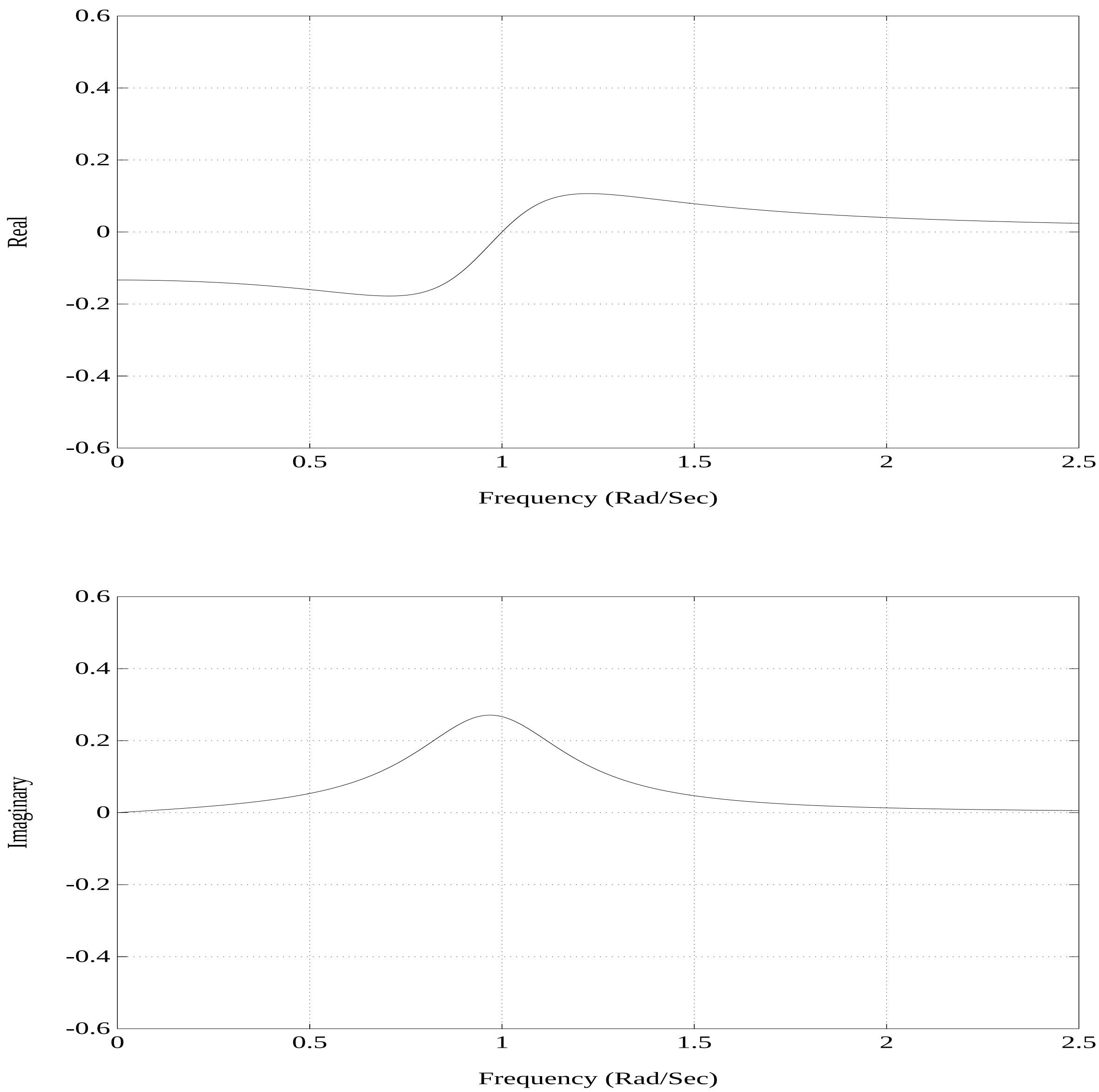


**Figure 5-4.** FRF, 2 DOF System, Proportionally Damped

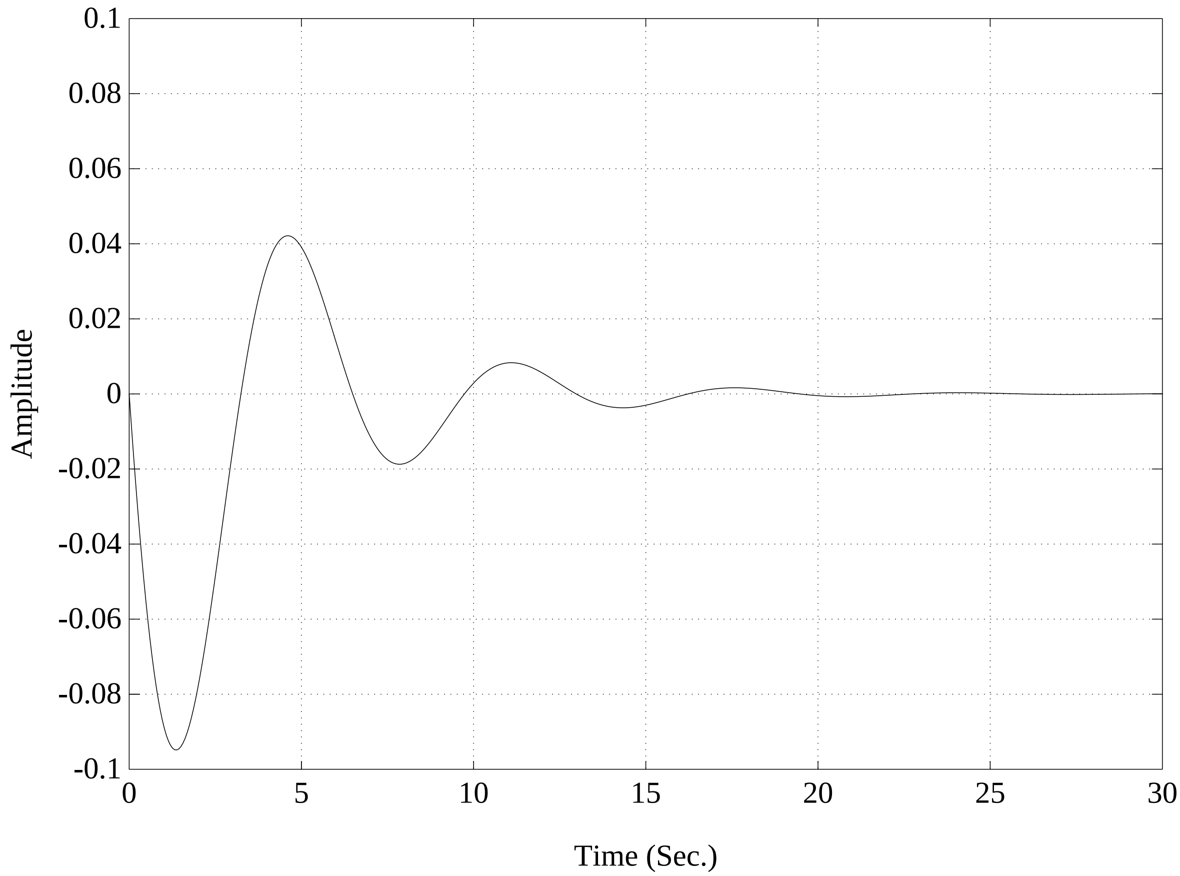


**Figure 5-5.** IRF, 2 DOF System, Proportionally Damped

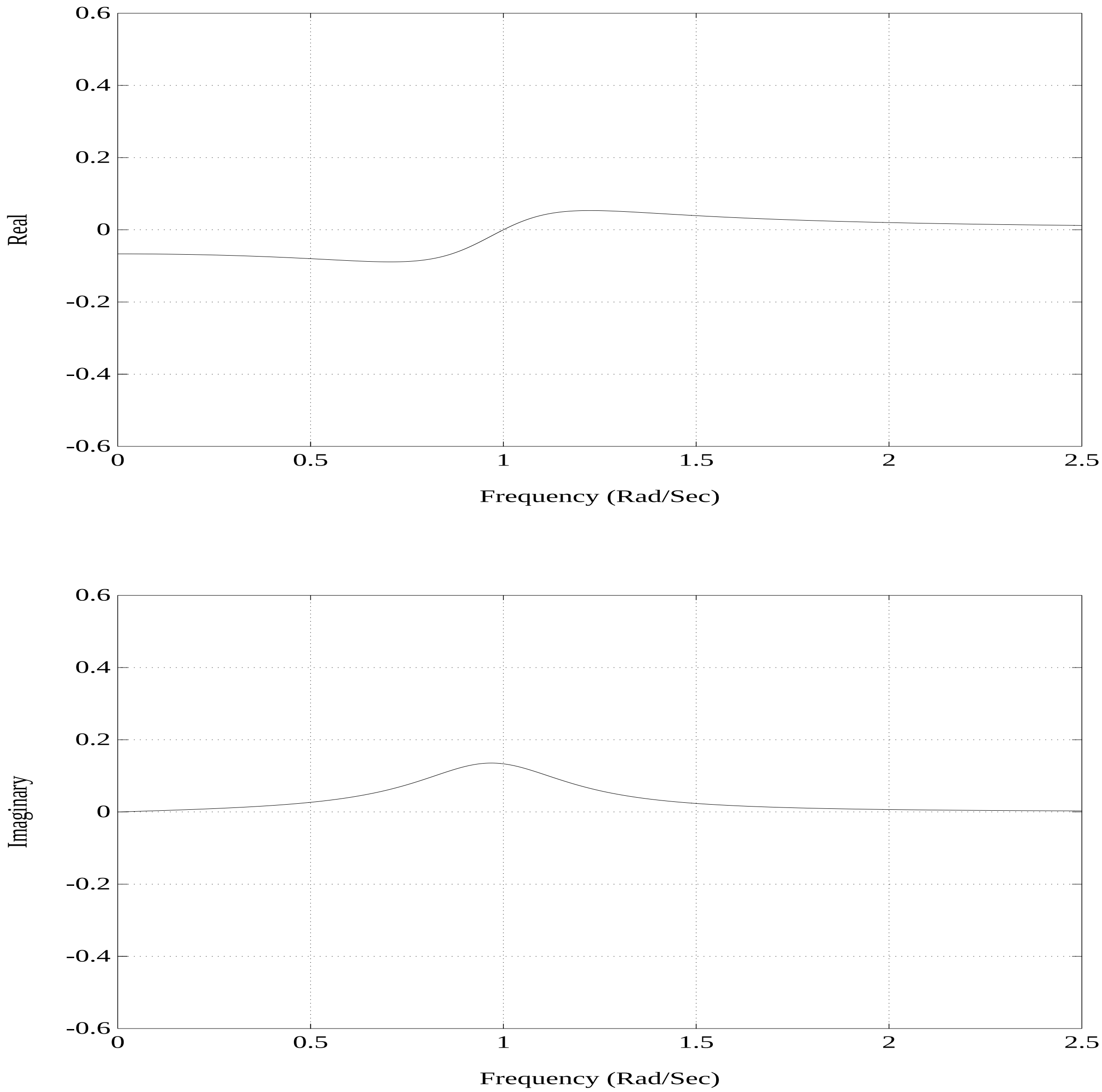


**Figure 5-6.** FRF, 2 DOF System, Proportionally Damped, First Mode

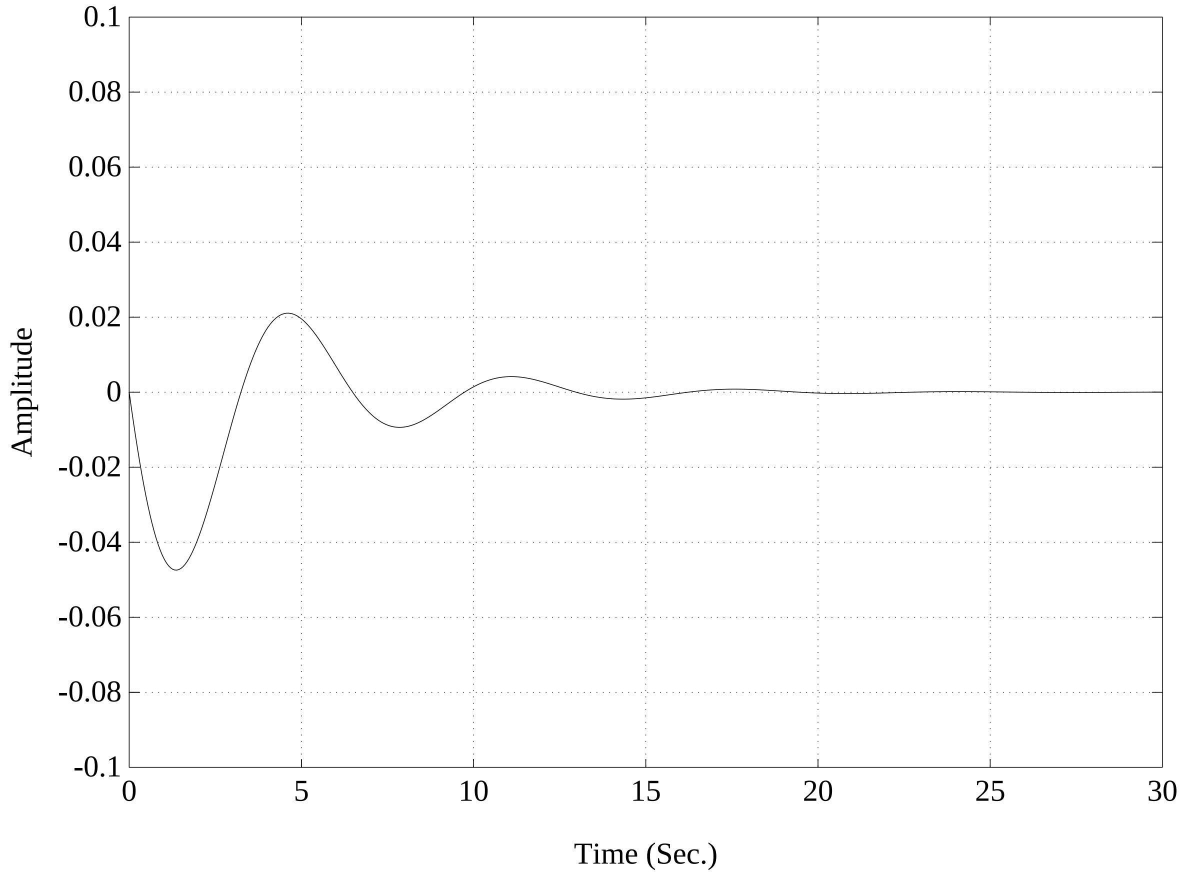


**Figure 5-7.** IRF, 2 DOF System, Proportionally Damped, First Mode

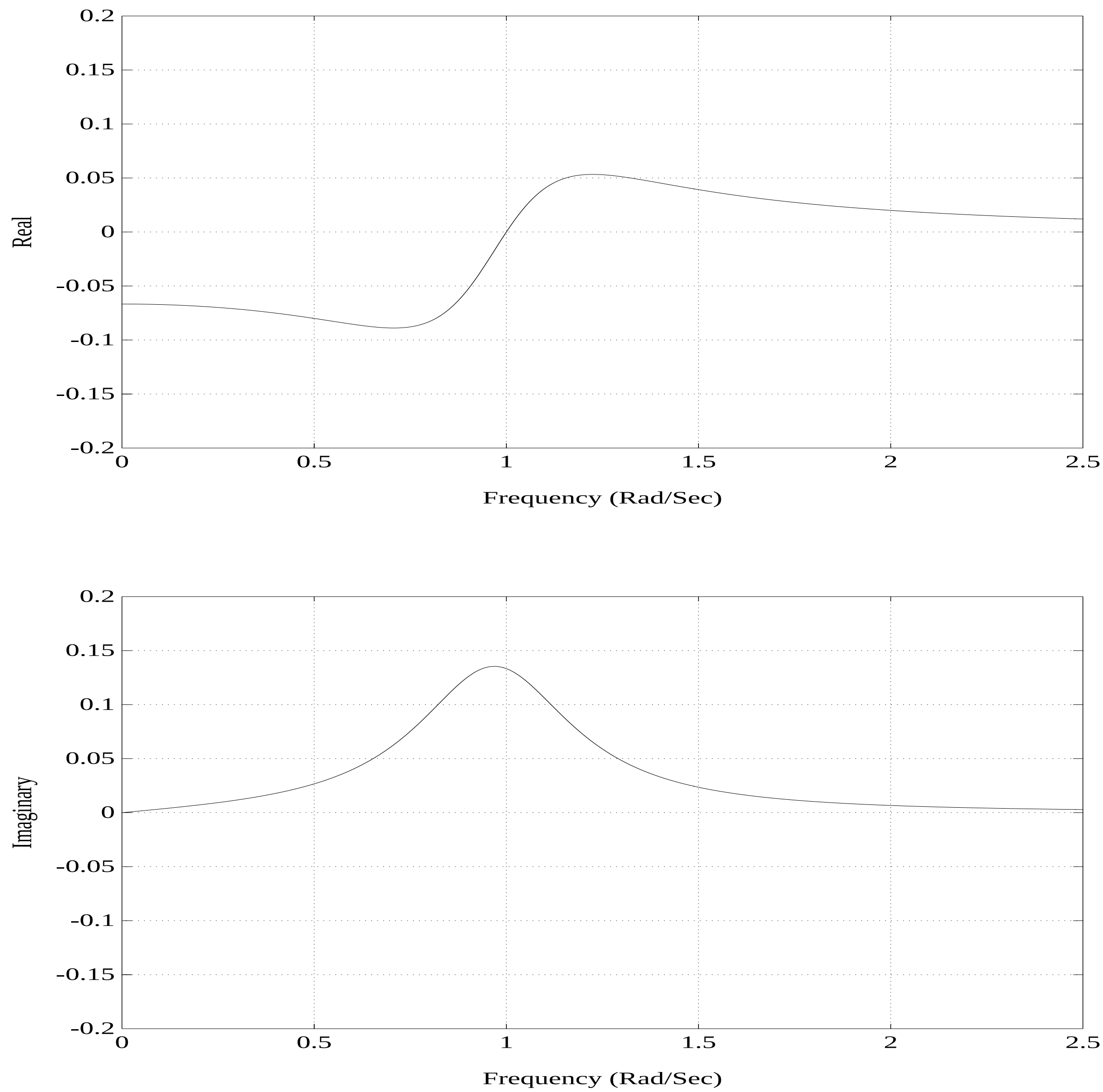


**Figure 5-8.** FRF, 2 DOF System, Proportionally Damped, Second Mode

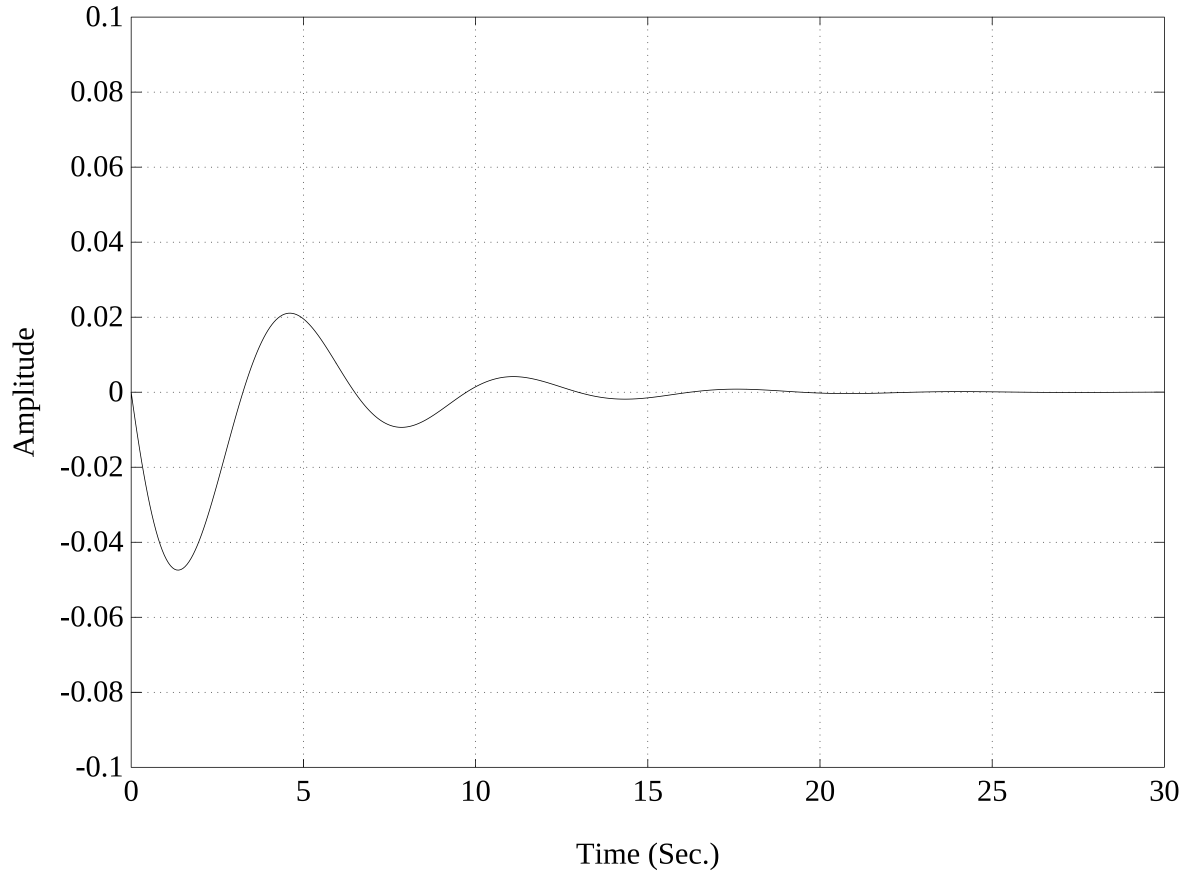


**Figure 5-9.** IRF, 2 DOF System, Proportionally Damped, Second Mode

## 5.4 Non-Proportional Damping Example

The same two degree of freedom system, in terms of mass and stiffness, as before will now be used to explain the effect of non-proportional damping.

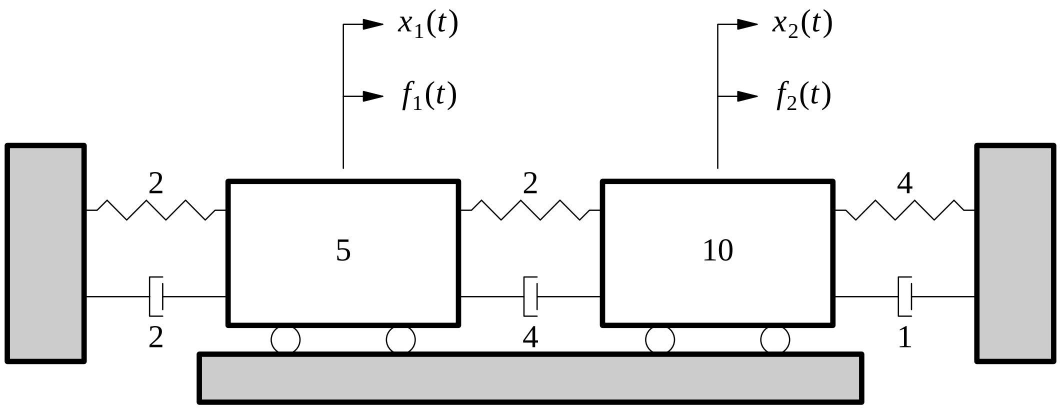


**Figure 5-10.** Two Degree of Freedom System

The damping coefficients have been chosen such that they are not proportional to the mass and/or stiffness. Using Equation 5.22, the transfer function matrix of the resulting system is:

$$[\ H(s)\ ] = \frac{\begin{bmatrix} 10s^2 + 5s + 6 & 4s + 2 \\ 4s + 2 & 5s^2 + 6s + 4 \end{bmatrix}}{50[s^4 + (17/10)\ s^3 + (42/25)\ s^2 + (4/5)\ s + 2/5]} \tag{5.31}$$

The poles of the transfer function are the roots of the characteristic equation:

$$s^4 + \frac{17}{10}\ s^3 + \frac{42}{25}\ s^2 + \frac{4}{5}\ s + \frac{2}{5} = 0 \tag{5.32}$$

The roots are:

$$\lambda_1 = -\,0.095363 + j\;0.629494 \;\;(\,rad/sec\,)$$
$$\lambda_1^* = -\,0.095363 - j\;0.629494 \;\;(\,rad/sec\,)$$

$$\lambda_2 = -\,0.754635 + j\;0.645996 \;\;(\,rad/sec\,)$$
$$\lambda_2^* = -\,0.754635 - j\;0.645996 \;\;(\,rad/sec\,)$$

These poles are the damping and frequency parameters for the system's two modes of vibration. Recall: $\lambda_r = \sigma_r + j\;\omega_r$.

As before, the modal vectors of the system can be determined by evaluating one row or column of the transfer function matrix in terms of partial fractions. In a previous example, the first column of the transfer function matrix was utilized in order to determine the modal vectors. This time, to illustrate the pervasiveness of the modal vectors with respect to all elements of $[\,H\,(s)\,]$, the last row of the transfer function matrix will be utilized. That is:

$$H_{21}\,(\mathrm{s}) \;\;\text{and}\;\; H_{22}\,(\mathrm{s})$$

$H_{21}$ (s) can be measured by exciting the system at mass 1 and measuring the response at mass 2. Similarly, $H_{22}$ (s) can be measured by exciting the system at mass 2 and measuring the response at mass 2. $H_{22}$(s) and $H_{21}$ (s) can now be expanded in terms of partial fractions.

$$H_{22}\,(s) = \frac{5s^2 + 6s + 4}{50\,(s-\lambda_1)\,(s-\lambda_1^*)\,(s-\lambda_2)\,(s-\lambda_2^*)}$$

After some work:

$$H_{22}\,(s) = \frac{0.003707 - j\;0.058760}{(s-\lambda_1)} + \frac{0.003707 + j\;0.058760}{(s-\lambda_1^*)}$$

$$+\,\frac{-0.003707 - j\;0.016358}{(s-\lambda_2)} + \frac{-0.003702 + j\;0.016358}{(s-\lambda_2^*)}$$

Also:

$$H_{21}\,(s) = \frac{4s^2 + 2}{50\,(s-\lambda_1)\,(s-\lambda_1^*)\,(s-\lambda_2)\,(s-\lambda_2^*)}$$

After some more work:

$$H_{21}\,(s) = \frac{-\,0.003473 - j\;0.050100}{(s-\lambda_1)} + \frac{-\,0.003473 + j\;0.050100}{(s-\lambda_1^*)}$$

$$+\;\frac{0.003473 + j\;0.045270}{(s-\lambda_2)} + \frac{0.003473 - j\;0.045280}{(s-\lambda_2^*)}$$

Using the residues as the modal vectors directly (without any normalization or scaling), the following vectors result:

Mode 1: $\lambda_1$ = - 0.095363 + $j$ 0.629494

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 = \begin{Bmatrix} -\,0.003473 - j\;0.050100 \\ +\,0.003707 - j\;0.058760 \end{Bmatrix}$$

Mode 2: $\lambda_2$ = - 0.754635 + $j$ 0.645996

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_2 = \begin{Bmatrix} +\,0.003473 + j\;0.045270 \\ -\,0.003707 - j\;0.016358 \end{Bmatrix}$$

The modal vectors are obviously no longer purely real. In both modal vectors, regardless of how the modal vectors are normalized, the description of the modal vector will now require a vector of complex numbers. This will in general be true for any system with non-proportional damping.

The above modal vectors can be converted to amplitude and phase.

Thus, for modal vector one:

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.050220, & -\,93.96^\circ \\ 0.058877, & -\,86.39^\circ \end{Bmatrix}_1$$

For modal vector two:

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_2 = \begin{Bmatrix} 0.045403, & +85.61° \\ 0.016765, & -102.77° \end{Bmatrix}_2$$

For mode one, as in the previous examples, the two masses are moving approximately in phase. Likewise, for mode two, as in the previous examples, the two masses are moving approximately out of phase. Thus, the non-proportionally damped system, which resulted in complex mode shapes, has phase terms which are not generally 0° or 180°.

Another way of comparing the differences between real modal vectors and complex modal vectors is as follows: A real modal vector represents a mode of vibration where all the points on the structure pass through their equilibrium positions at the same time. A complex modal vector represents a mode of vibration where the points tested do not pass through their equilibrium positions at the same time. In other words, a real mode appears as a standing wave and a complex mode appears as a traveling wave with respect to the structure. The nodal point or line for real modes will be stationary with respect to the structure. Conversely, the nodal point or line for a complex mode will be non-stationary with respect to the structure.

Many structures tested will have nearly real mode shapes, that is, the phase term will be very close to being 0° or 180°. Structures in general can have both real and complex mode shapes. In the previous example, the first modal vector is within ± 4° of being totally real, whereas the second modal vector is within ± 13° of being real. Thus, the second modal vector is more complex than the first modal vector.

### 5.4.1 Impulse Response Function Example

The impulse response function is computed in a similar fashion as for the proportionally damped system. From Equation 5.27, the basic definition of the impulse response function is:

$$H_{pq}(t) = \mathbf{L}^{-1} \left\{ \sum_{r=1}^{N} \frac{A_{pqr}}{(s-\lambda_r)} + \frac{A^*_{pqr}}{(s-\lambda^*_r)} \right\} \tag{5.33}$$

The only difference now is that the residues $A_{pqr}$ are no longer purely imaginary but are, in general, complex. For a two-degree of freedom system, the impulse response function becomes:

$$h_{pq}\,(t) = A_{pq1}\, e^{\,\lambda_1\, t} + A^*_{pq1}\, e^{\lambda^*_1\, t} + A_{pq2}\, e^{\lambda_2\, t} + A^*_{pq2}\, e^{\lambda^*_2\, t} \tag{5.34}$$

The only difference between the impulse response function of a system with complex modal vectors (Equation 5.34) and a system with real modal vectors (Equation 5.29 and 5.30) is that the individual contributions from each mode of vibration to the total impulse response function do not begin with a value of zero response at time zero. Note that the sum of these individual contributions must total zero at time zero for the system to be causal. Again note that the residues, which are proportional to the product of the modal coefficients of the input and response degrees of freedom, appear as the amplitude of the damped sinusoids.

A typical frequency response function is once again illustrated in Figure (5-11), in terms of its real and imaginary parts. This frequency response function represents the response of the system at degree of freedom 1 due to a force applied to the system at degree of freedom 2 $H_{12}(\omega)$. Figure (5-12) is the equivalent impulse response function of Figure (5-11). Likewise, this impulse response function represents the response of the system at degree of freedom 1 due to a unit impulse applied to the system at degree of freedom 2 $h_{12}(t)$. Figure (5-11) depicts a frequency response function of a two degree of freedom system that has complex modal vectors. The only difference between this plot and that of Figure (5-4) is that the residues in this case have a phase angle of other than 0° or 180°. The impulse response function of this system is shown in Figure (5-11). Notice that the impulse response function does start at zero for t=0. This is always the case for any causal system with or without complex modal vectors. Notice, though, that the impulse response function contribution from each individual mode does not start at zero for this case of complex modal vectors. Nevertheless, for a causal system, the sum of the residues, and therefore the combination of magnitudes and phase angles in Equation 5.27, will always yield an impulse response function that is zero at time zero. (The sum of the residues will be equal to zero for any system where the order of the numerator polynomial is two or more less than the order of the denominator polynomial.) As before, the frequency response function can be represented as (Figure (5-11)) is the sum of two single degree of freedom systems (Figures (5-13) and (5-15)). Another indication of a complex modal vector is the asymmetric characteristic of the imaginary part of the frequency response function for such a simple system. In Figure (5-6) and (5-8), note that the imaginary parts are symmetrical, whereas in Figures

(5-13) and (5-15) the imaginary parts are not symmetrical. The degree to which they are not symmetrical is an indication of how complex that particular component of the modal vector is. Figures (5-14) and (5-16) are the impulse response functions of Figures (5-13) and (5-15).

The modal vectors and modal frequencies of the undamped, proportionally damped, and non-proportionally damped system can now be compared for the simple two degree of freedom system used in all the examples. Remember that the stiffnesses and masses are the same throughout the examples. Therefore, the modal vectors and modal frequencies can only be different as a result of the damping elements. For comparison purposes, the unscaled residues from the standard form of the transfer function will be used as the modal vectors for each case.

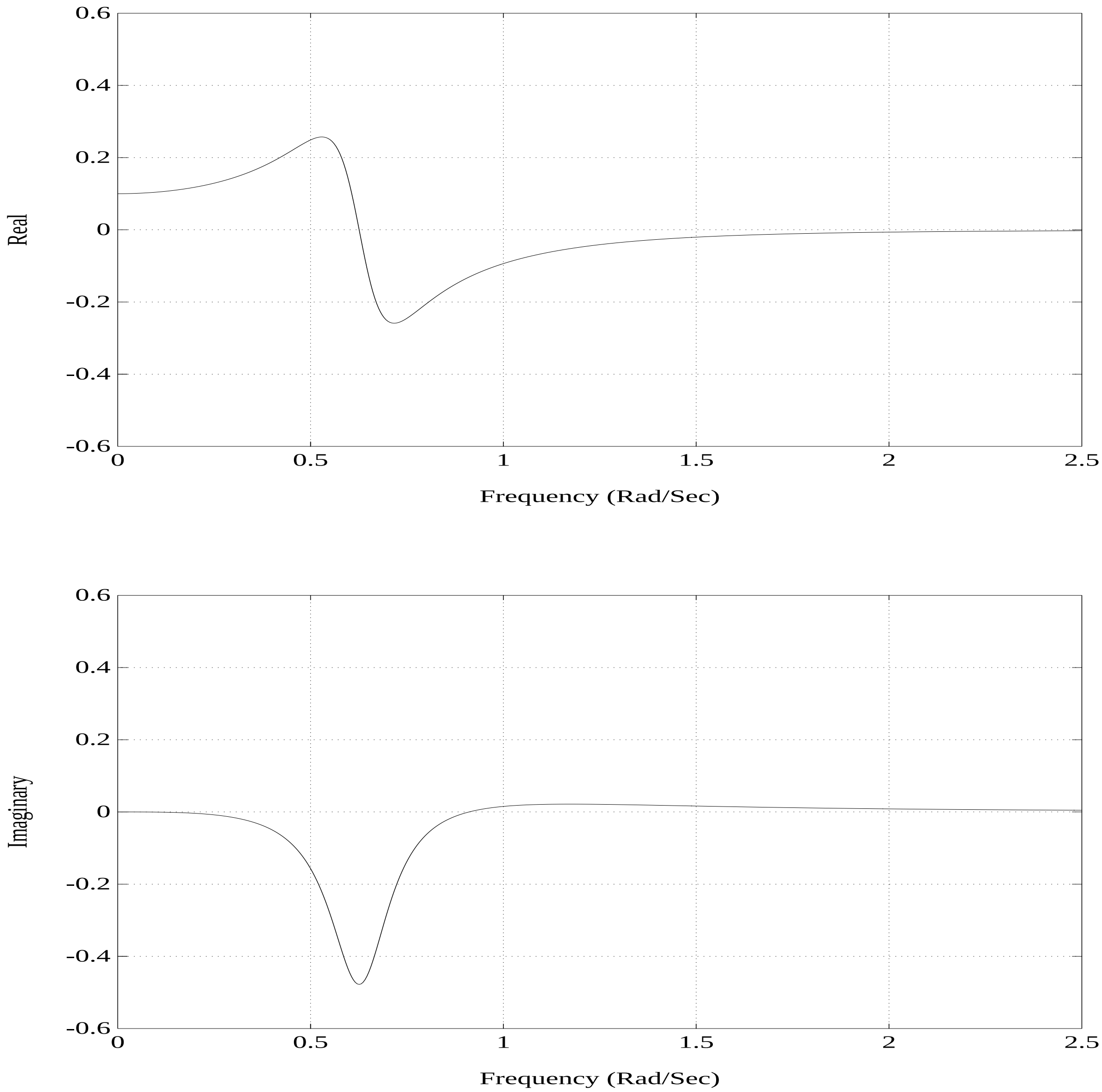


**Figure 5-11.** FRF, 2 DOF System, Non-Proportionally Damped

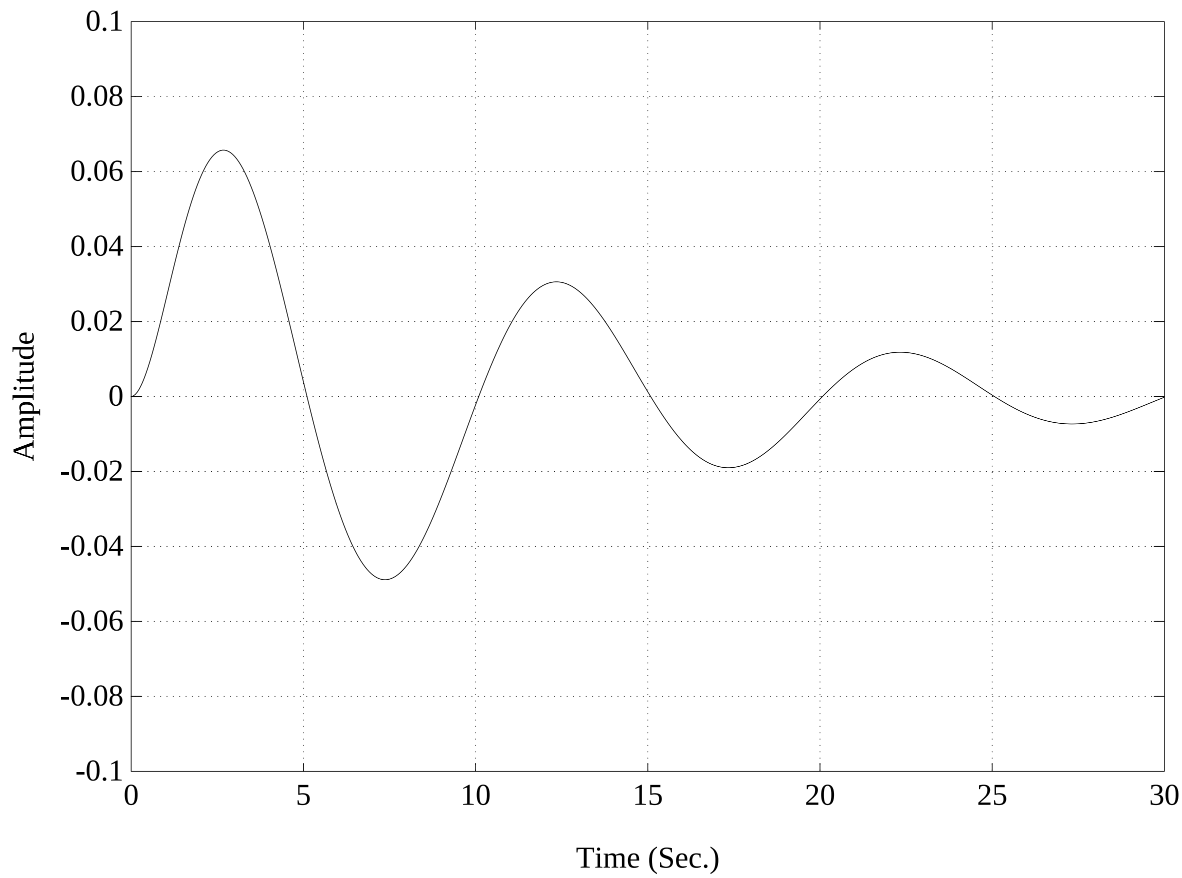


**Figure 5-12.** IRF, 2 DOF System, Non-Proportionally Damped

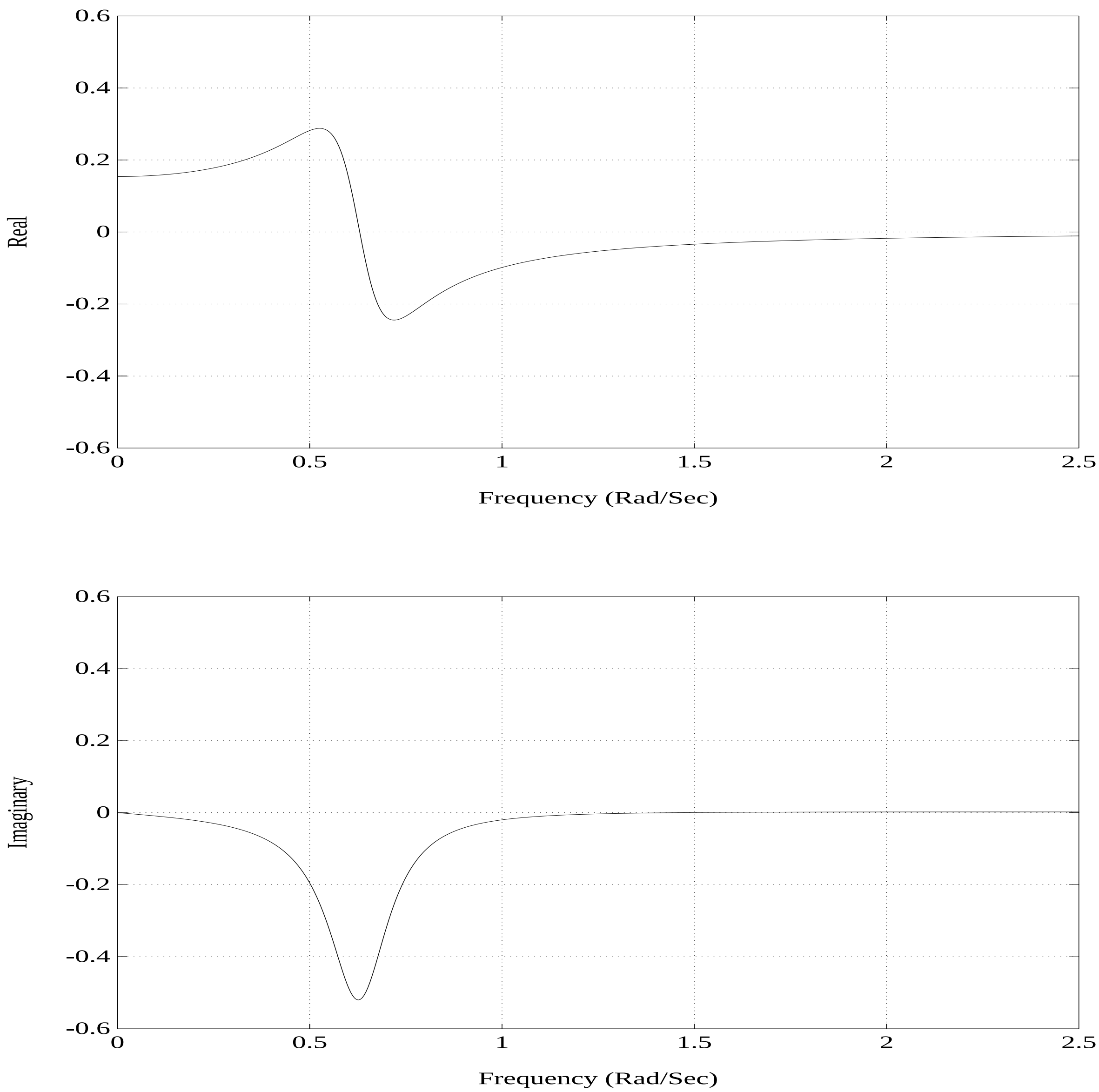


**Figure 5-13.** FRF, 2 DOF System, Non-Proportionally Damped, First Mode

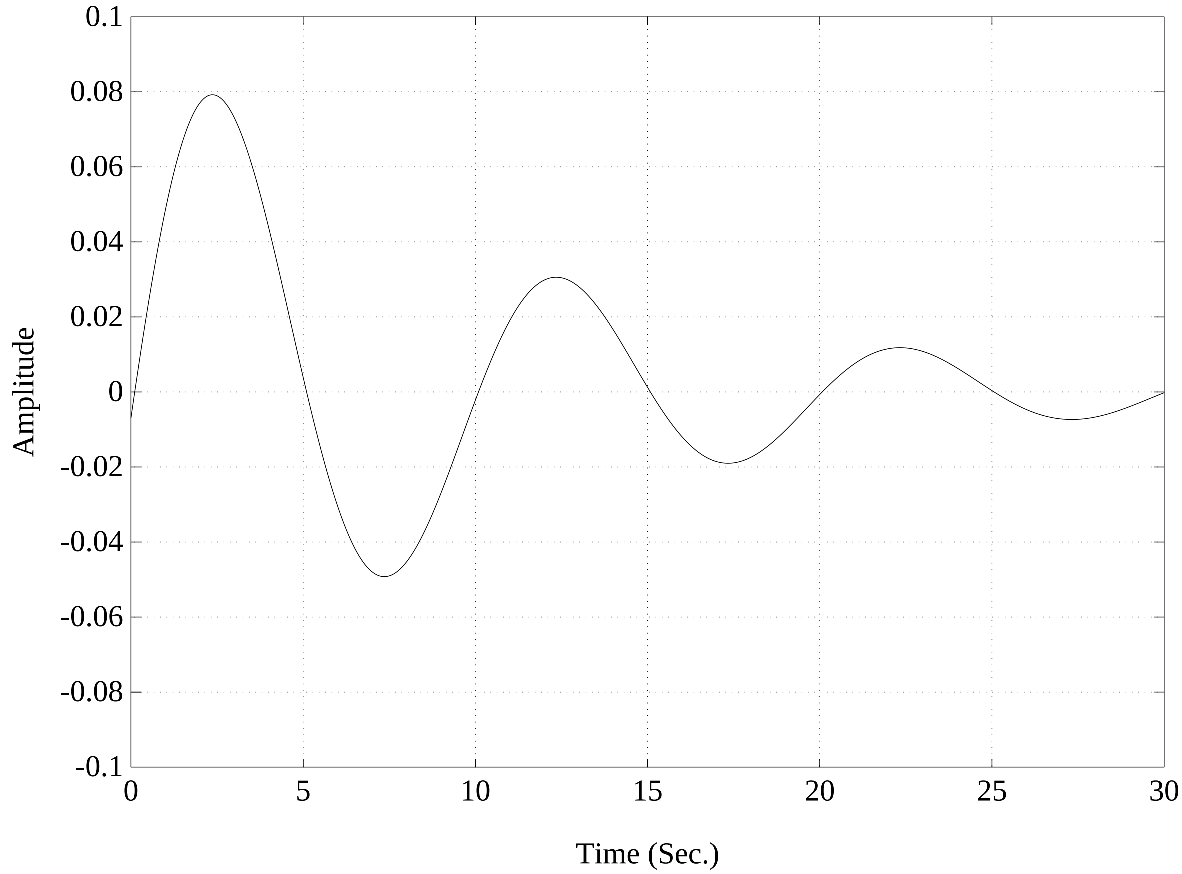


**Figure 5-14.** IRF, 2 DOF System, Non-Proportionally Damped, First Mode

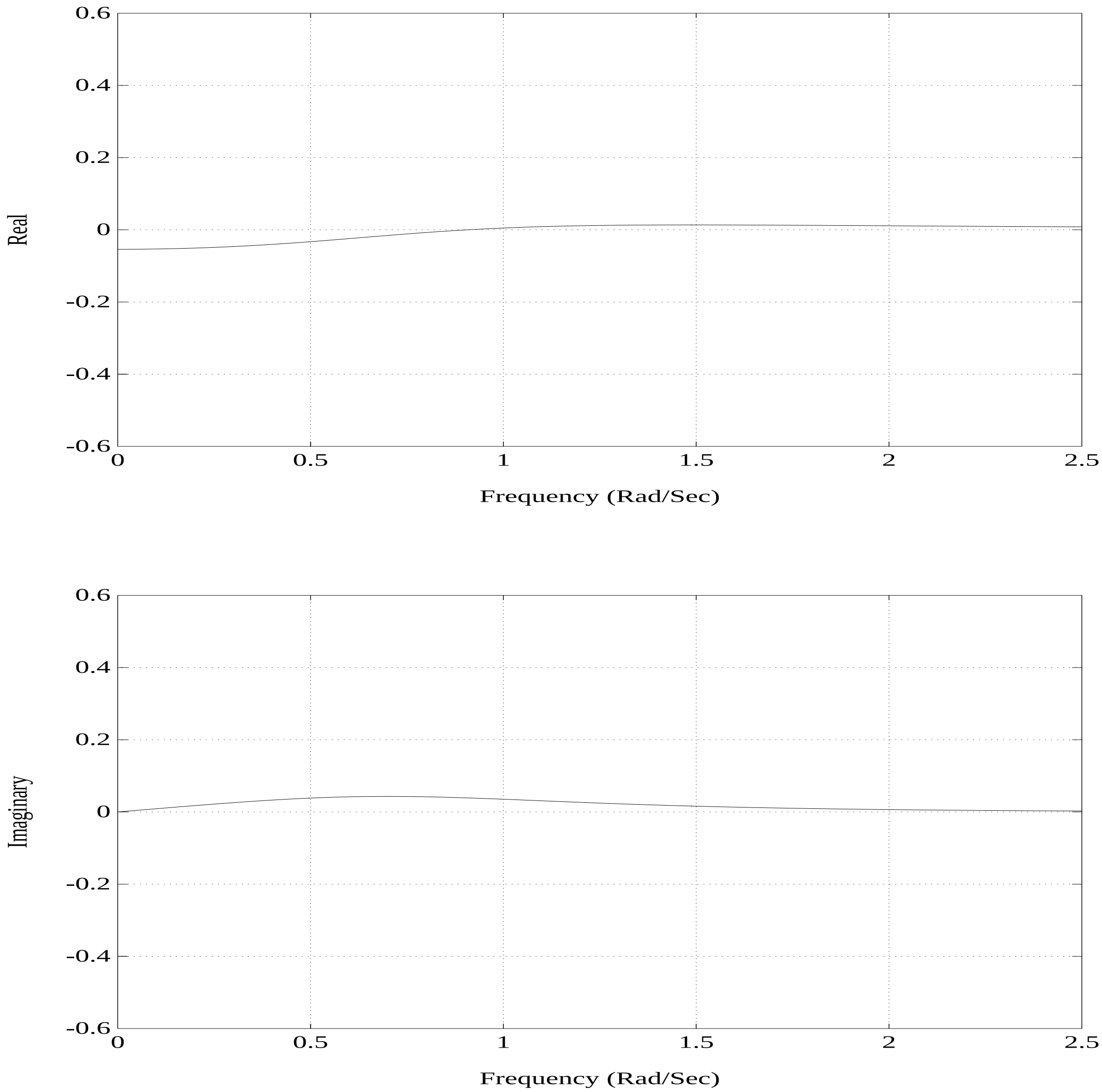


**Figure 5-15.** FRF, 2 DOF System, Non-Proportionally Damped, Second Mode

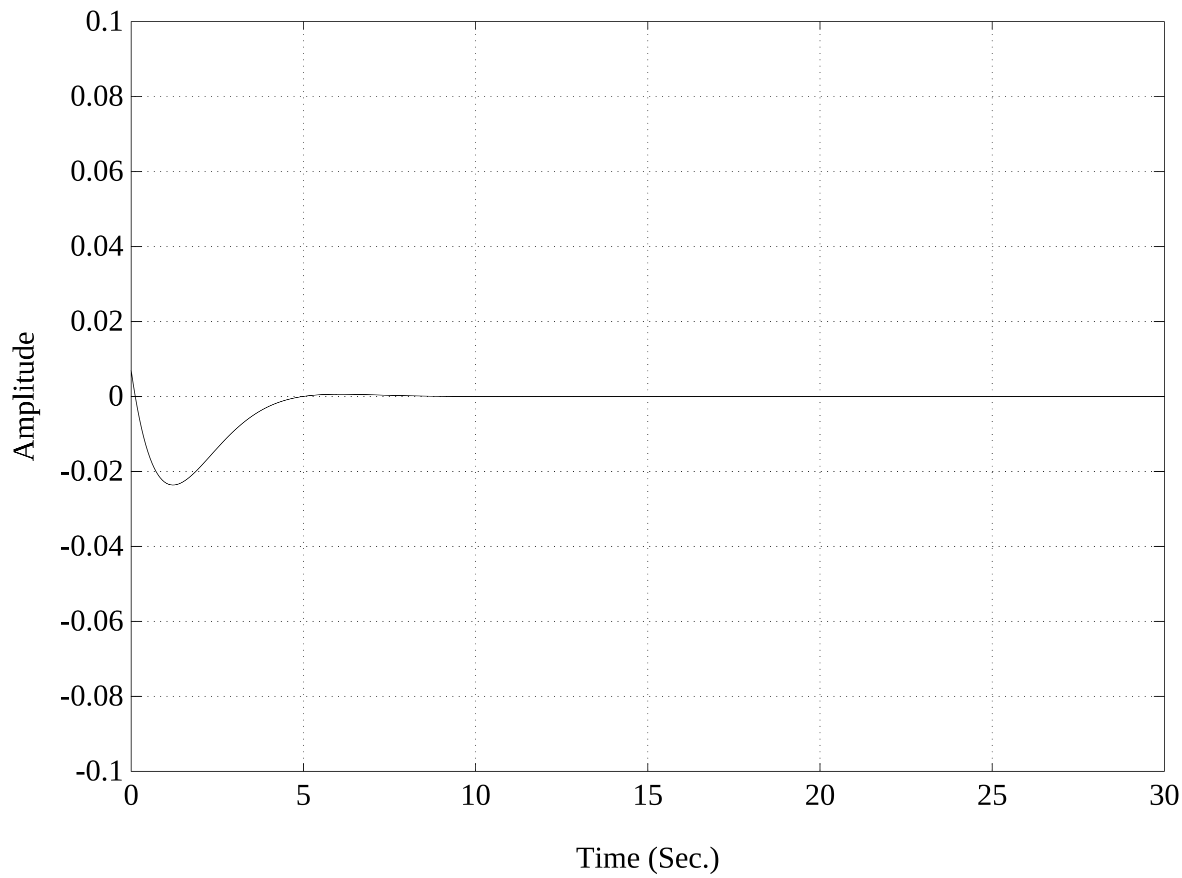


**Figure 5-16.** IRF, 2 DOF System, Non-Proportionally Damped, Second Mode

## 5.5 Summary of Modal Parameters (Undamped and Damped Systems)

Recalling the modal vectors from the previous examples:

Mode 1 | Mode 2

Undamped Case:

$$\lambda_1 = j\,\sqrt{2/5} \qquad \lambda_2 = j$$

$$\psi_1 = \begin{Bmatrix} -j\,\dfrac{\sqrt{2/5}}{12} \\ -j\,\dfrac{\sqrt{2/5}}{12} \end{Bmatrix} \qquad \psi_2 = \begin{Bmatrix} -j\,\dfrac{1}{15} \\ j\,\dfrac{1}{30} \end{Bmatrix}$$

Proportionally Damped Case:

$$\lambda_1 = \frac{1}{10} + j\,(\frac{\sqrt{39}}{10}) \qquad \lambda_2 = -\frac{1}{4} + j\,(\frac{\sqrt{15}}{4})$$

$$\psi_1 = \begin{Bmatrix} -j\,\dfrac{\sqrt{39}}{117} \\ -j\,\dfrac{\sqrt{39}}{117} \end{Bmatrix} \qquad \psi_2 = \begin{Bmatrix} -j\,\dfrac{4\sqrt{15}}{225} \\ j\,\dfrac{2\sqrt{15}}{225} \end{Bmatrix}$$

Non-proportionally Damped Case:

$$\lambda_1 = -0.095363 + j\,0.629494 \qquad \lambda_2 = -0.754635 + j\,0.645996$$

$$\psi_1 = \begin{Bmatrix} -0.008569 - j\,0.041970 \\ -0.003473 - j\,0.050100 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} 0.008569 - j\,0.122700 \\ 0.003473 + j\,0.045270 \end{Bmatrix}$$

Note that for the non-proportional damped case, the residues given in the table are for the same frequency response functions used in the undamped and proportionally damped cases (the first column of the frequency response function matrix). This required the computation of the residues for $H_{11}\,(s)$.

Recall that a pole ($\lambda_r$) may be written as:

$$\lambda_r = \sigma_r + j\ \omega_r = (-\ \zeta_r + j\ \sqrt{1-\zeta_r^2})\ \Omega_n \tag{5.35}$$

where:

- $\zeta_r$ = damping ratio
- $\Omega_r$ = undamped natural frequency
- $\sigma_r$ = damping factor
- $\omega_r$ = damped natural frequency

From Equation 5.35, the following relationships can also be noted:

$$\zeta_r = \frac{-\ \sigma_r}{\sqrt{\omega_r^2 + \sigma_r^2}}$$

$$\Omega_r = \frac{-\ \sigma_r}{\zeta_r}$$

The first modal vector for the 3 different cases can now be compared. Note that the modal vectors have been converted to amplitude and phase.

For modal vector one ( $\lambda_1$ ):

| Undamped | Prop. Damped | Non-prop. Damped |
| --- | --- | --- |
| $\begin{Bmatrix} 0.052705, -90° \\ 0.052705, -90° \end{Bmatrix}_1$ | $\begin{Bmatrix} 0.053376, -90° \\ 0.053376, -90° \end{Bmatrix}_1$ | $\begin{Bmatrix} 0.042836, -101.53° \\ 0.050220, -93.96° \end{Bmatrix}_1$ |
| $\Omega_1 = 0.632456$ | $\Omega_1 = 0.632456$ | $\Omega_1 = 0.63667$ |
| $\zeta_1 = 0\%$ | $\zeta_1 = 15.8114\%$ | $\zeta_1 = 14.9782\%$ |

For modal vector two ( $\lambda_2$ ):

| Undamped | Prop. Damped | Non-prop. Damped |
| --- | --- | --- |
| $\begin{Bmatrix} 0.066667, -90° \\ 0.033333, +90° \end{Bmatrix}_2$ | $\begin{Bmatrix} 0.068853, -90° \\ 0.034427, +90° \end{Bmatrix}_2$ | $\begin{Bmatrix} 0.122999, -86.00° \\ 0.045403, +85.61° \end{Bmatrix}_2$ |
| $\Omega_2 = 1.0$ | $\Omega_2 = 1.0$ | $\Omega_2 = 0.99337$ |
| $\zeta_2 = 0\%$ | $\zeta_2 = 25\%$ | $\zeta_2 = 75.9671\%$ |

In conclusion, note that the addition of damping (proportional and non-proportional) has not affected the undamped natural frequencies for either modal vector. This will always be the case. Furthermore, the overall behavior of the modal vectors remained the same. That is, in the first mode the 2 masses are moving in the same direction and in the second mode they are moving in opposite directions. The major difference arises in the phase of the mode shapes. Note that for the non-proportionally damped system, the phase of the components is other than 0° or 180°. These types of modes are again referred to as complex modes. How much the phase differs from 0° or 180° is directly related to the systems mass, damping, and stiffness distributions. Note that as the phase approaches 0° or 180° for a particular mode, the modal vector can be approximated as a real or normal mode by setting the phase equal to 0° or 180°.