6. FREQUENCY RESPONSE FUNCTION SYNTHESIS

In this section, the approach for constructing the entire frequency response function matrix [$H(\omega)$] will be explained. The frequency response function matrix [$H(\omega)$], which is made up of $N_0 \times N_i$ frequency response functions, can be synthesized by using the data from N_o frequency response functions where N_o is the number of measured response degrees of freedom (physical response points times number of direction(s) at each physical point) on the structure. For the following discussion, N_o is assumed to be larger than N_i and N_o is assumed to include the N_i measurement degrees of freedom. For the case of fixed response degrees of freedom with a large number of applied inputs (impact testing, for example), N_i will be much larger than N_o but since the [$H(\omega)$] is reciprocal, the same assumption can be made concerning N_i. These are reasonable assumptions for all testing situations. There are two restrictions that apply to being able to synthesize the complete matrix accurately from only N_o or N_i frequency response functions. First of all, the N_o or N_i frequency response functions must consist of either a complete row or complete column of frequency response function measurements from the frequency response function matrix. Secondly, the N_o or N_i frequency response functions must contain non-zero residue information for every modal vector present in the structure within the frequency range of interest. This means that if the modal vector is entirely zero (due to excitation at the node of a modal vector) proper frequency response function synthesis will not be possible.

Recall the general form of the frequency response function matrix [$H(\omega)$] (Equation 4.25) in terms of partial fractions.

$$[\ H(\omega)\ ] = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) & . & . & H_{1m}(\omega) \\ H_{21}(\omega) & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ H_{m1}(\omega) & . & . & . & H_{mm}(\omega) \end{bmatrix} \tag{6.1}$$

Assuming that only the k-th column of [$H(\omega)$] has been measured by way of frequency response functions, the k-th column can be represented as follows:

$$\left\{ \begin{array}{l} H_{1k}\ (\ \omega\ ) \\ H_{2k}\ (\ \omega\ ) \\ \cdot \\ \cdot \\ H_{ik}\ (\ \omega\ ) \\ \cdot \\ \cdot \\ H_{mk}\ (\ \omega\ ) \end{array} \right\} = \sum_{r=1}^{N} \frac{\{A_r\}_k}{(j\ \omega - \lambda_r)} + \frac{\{A_r^*\}_k}{(j\ \omega - \lambda_r^*)} \tag{6.2}$$

where:

- m = the number of measurements (N_o or N_i).

Equation 6.2 is the mathematical description of the N_o frequency response function measurements that were obtained during the test of the structure. For a particular mode of vibration, the N modal frequencies and the N residues for each of the N_o frequency response functions can be determined through the use of a modal parameter estimation algorithm.

Once these modal parameters are known, frequency response functions from column k or any other column in the frequency response function can be synthesized using a partial fraction model with the correct residues for the particular input and output degrees of freedom desired. The synthesis is formulated on a frequency by frequency basis for each mode - the response for each mode is summed together to get the total frequency response function as is shown in the following figures:

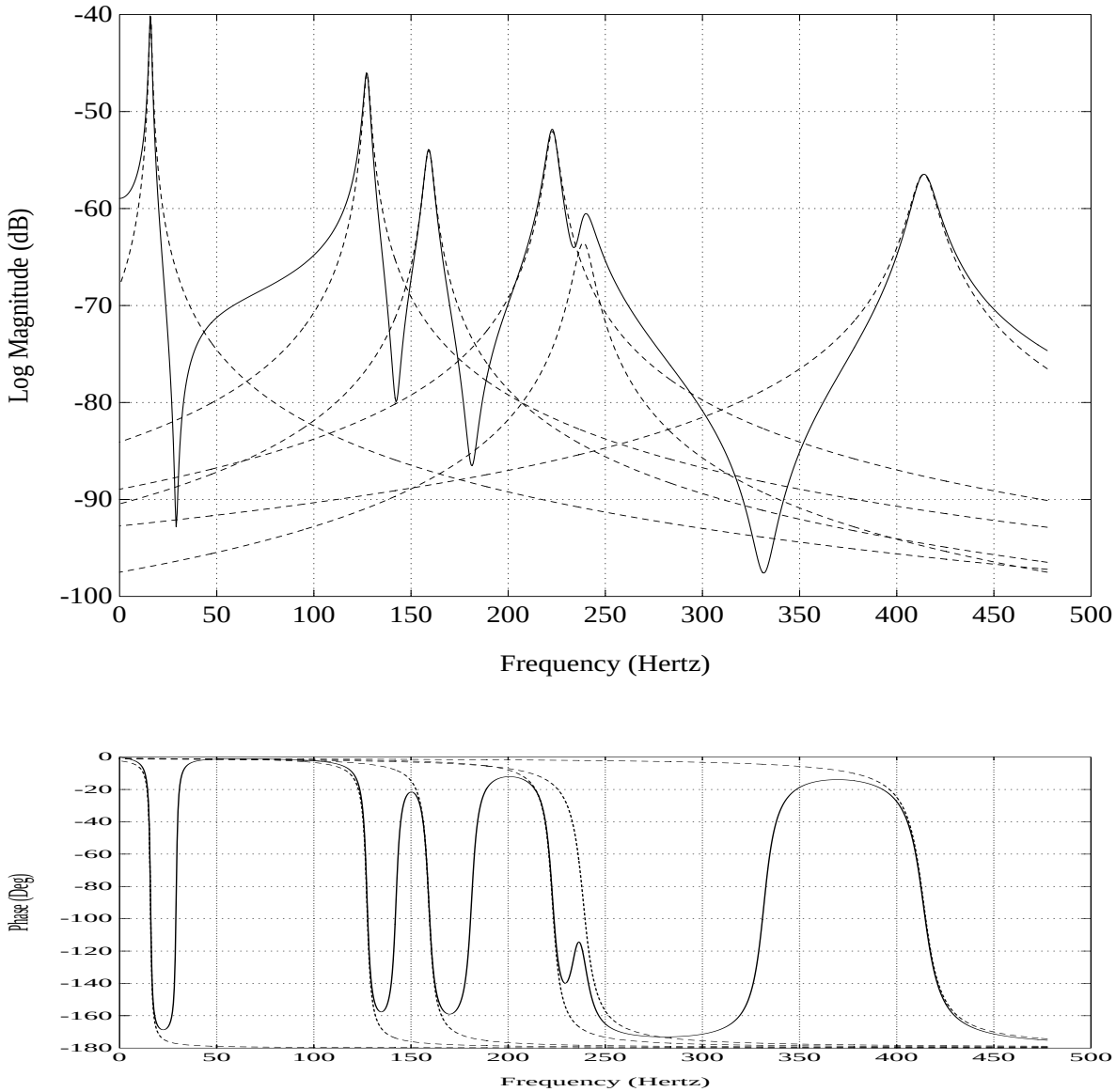

Figure 6-1. Frequency Response Function Synthesis

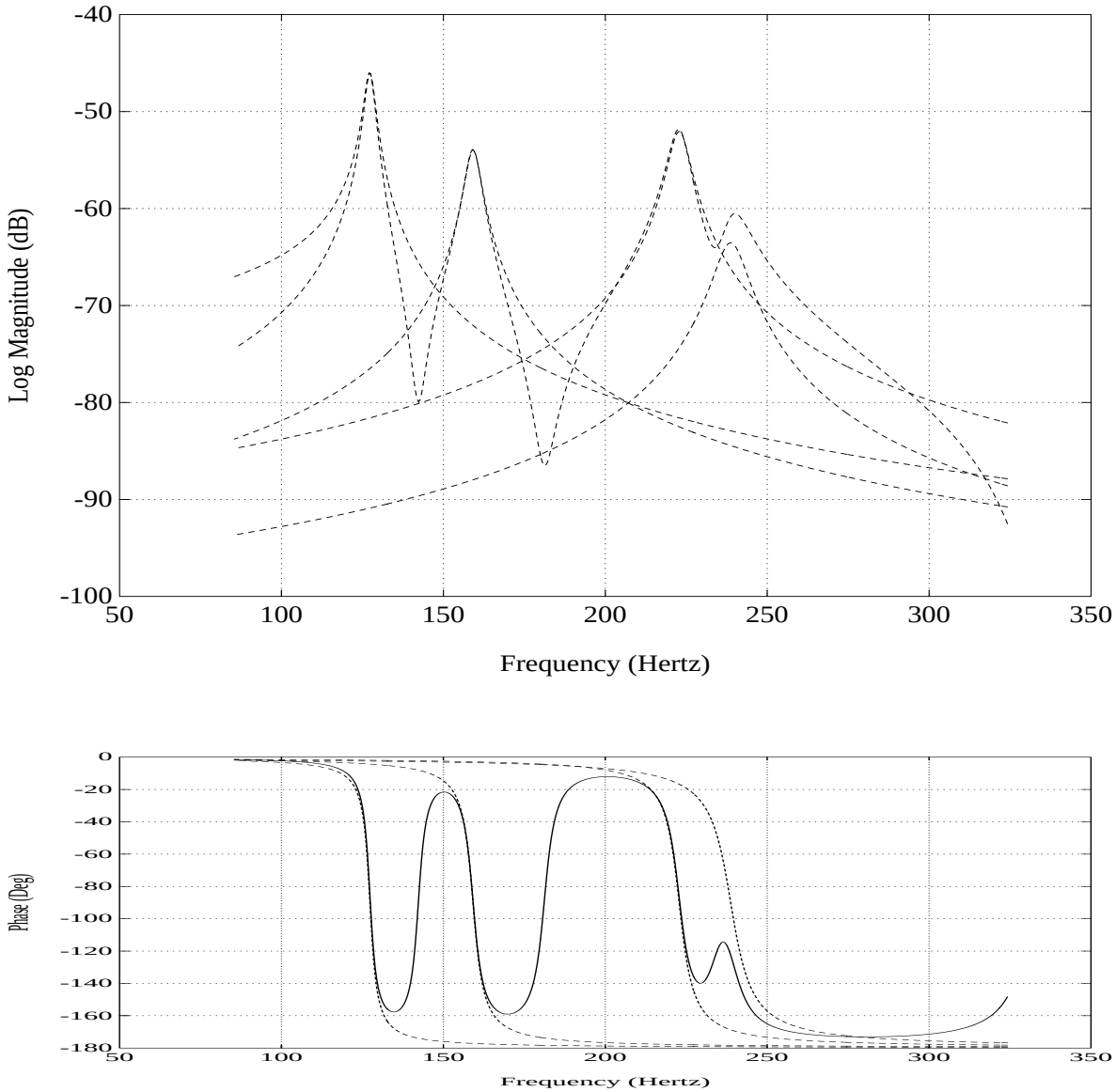

Figure 6-2. Frequency Response Function Synthesis

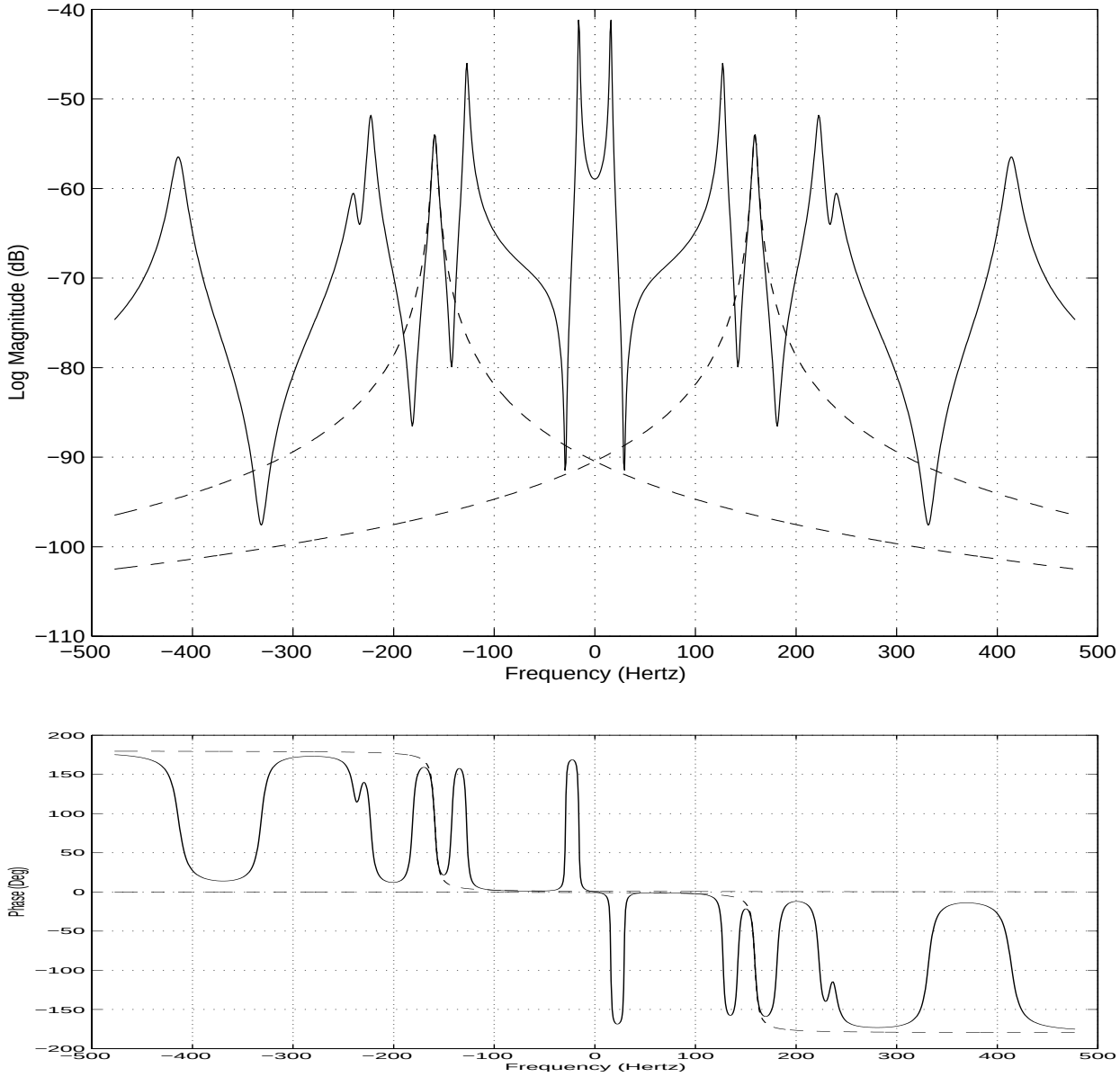

Figure 6-3. Frequency Response Function Synthesis

In order to construct all of the elements that make up [$H(\omega)$], all of the elements of $[A]_r$ for each mode will be required. Previously, the residue matrix $[A]_r$ for a particular mode has been shown to be directly related to the modal vector according to the following equation:

$$\left[\ A_r\ \right] = Q_r\ \{\ \psi\ \}_r\ \{\ \psi\ \}_r^T = Q_r \begin{bmatrix} \psi_1\ \psi_1 & \psi_1\ \psi_2 & \cdot & \cdot & \cdot & \cdot & \psi_1\ \psi_m \\ \psi_2\ \psi_1 & \psi_2\ \psi_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \psi_m\ \psi_1 & \psi_m\ \psi_2 & \cdot & \cdot & \cdot & \cdot & \psi_m\ \psi_m \end{bmatrix}_r \tag{6.3}$$

When the k-th column of [$H(\omega)$] has been measured, the k-th column of each residue matrix $[A]_r$ can be defined. Equation 6.3 can now be rewritten for only the k-th column of of the r-th mode of vibration.

$$\begin{Bmatrix} A_{1k} \\ A_{2k} \\ \cdot \\ \cdot \\ A_{ik} \\ \cdot \\ \cdot \\ A_{mk} \end{Bmatrix}_r = Q_r \begin{Bmatrix} \psi_1\psi_k \\ \psi_2\psi_k \\ \cdot \\ \cdot \\ \psi_i\psi_k \\ \cdot \\ \cdot \\ \psi_m\psi_k \end{Bmatrix}_r = Q_r\ \psi_{kr} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_i \\ \cdot \\ \cdot \\ \psi_m \end{Bmatrix}_r \tag{6.4}$$

where:

- m = Number of measurements (N_o or N_i).
- r = Mode number

In order to compute the residue for any position in the residue matrix for each mode of vibration, Equation 6.4 can be used to compute the values of Q_r and $\{\psi\}_r$. In order to do this, either Q_r or ψ_{kr} must be chosen according to a scaling criteria. At this point the individual elements of $\{\psi\}_r$ can now be found. Once these individual elements are known, any residue for a given mode can be computed using the following equation:

$$A_{pqr} = Q_r\ \psi_{pr}\ \psi_{qr} \tag{6.5}$$

If modal mass will be calculated, the individual values of Q_r and $\{\psi\}_r$ will be required. Therefore, Equation 6.5 can be used easily to synthesize any arbitrary residue. In reality, there is no need to know the individual values of Q_r and $\{\psi\}_r$ in order to synthesize another residue. If the k-th column of the residue matrix $[A]_r$ is again used to synthesize any arbitrary residue from another location in the residue matrix, the following equation is all that is required:

$$A_{pqr} = \frac{A_{pkr} \; A_{qkr}}{A_{kkr}} = \frac{Q_r \; \psi_{pr} \; \psi_{kr} \;\; Q_r \; \psi_{qr} \; \psi_{kr}}{Q_r \; \psi_{kr} \; \psi_{kr}} \tag{6.6}$$

Note that with Equation 6.6, any arbitrary residue can be synthesized if the proper elements of the frequency response function matrix and, therefore, the proper elements of each residue matrix $[A]_r$ have been measured. For example, the driving point information for the k-th column of any residue matrix need not be measured directly if the following elements have been measured:

$$A_{kkr} = \frac{A_{pkr} \; A_{qkr}}{A_{pqr}} \tag{6.7}$$

When the errors involved with measuring the driving point frequency response function are taken into account, Equation 6.7 may be a preferable way to estimate the driving point information even though this method requires information from another column of the residue matrix. Note that this will require that another excitation location be taken during the test of the structure.

In summary, one row or column of the frequency response function matrix, including driving point information, must be measured in order to synthesize any arbitrary measurement in the frequency response function matrix. A modal parameter estimation algorithm is then used on these measurements to determine the modal parameters. With the driving point residue information for a particular mode of vibration, any other required residue information can be synthesized by Equation 6.5 or Equation 6.6.

6.1 Displacement, Velocity and/or Acceleration Data

In practice, frequency response functions are often measured using an accelerometer to measure the response of the system and a load cell to measure the input to the system. Velocity or

displacement transducers could also be used to measure the response but have disadvantages with respect to size, configuration or datum. Regardless, the mathematical model that has been developed is a representation of displacement normalized to the force that caused the displacement. Note that if velocity normalized to force is actually measured, this data can be synthetically integrated to displacement over force by dividing the velocity over force measurement by $j\omega$. Likewise, if acceleration normalized to force is actually measured, this data can be synthetically integrated to displacement over force by dividing the acceleration over force measurement by $(j\omega)^2$. This approach has numerical problems near zero frequency but is generally satisfactory elsewhere.

6.2 Skewed Sensor Orientation

While it is common and somewhat desirable to align all sensors (input and output) in directions that are colinear with a set of orthogonal axes (x, y, z), this is not a requirement. Directional cosine information can be used to reorient the sensor information according to any desirable set of axes. Note that the spatial subscripts used to this point (pq for example) do not imply any specific constraint in this sense.

However, if modal scaling or the ability to synthsize arbitrary functions that may or may not have been measured is desired, the measurment of the driving point information is very important. If a skewed excitation is used with a set of essentially orthogonal response sensors, the simplest solution to this problem is to add a skewed response sensor, at the excitation location, so that a true driving point measurement can be estimated. With this in mind, the Equations 6.1 through 6.7 still apply when synthesis of residues and frequency response functions from data taken from a skewed input is the data represented by column k.

6.3 Simulation of Structural Response

Once all of the frequency response functions (transfer functions) for a particular system have been measured or synthesized, the response of the structure to any arbitrary set of inputs can be simulated or predicted. To illustrate the process, the two degree of freedom system that is represented by the following equation can be used as an example.

$$\left\{ \begin{array}{l} X_1(\omega) \\ X_2(\omega) \end{array} \right\} = \left[\begin{array}{ll} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{array} \right] \left\{ \begin{array}{l} F_1(\omega) \\ F_2(\omega) \end{array} \right\} \tag{6.8}$$

Equation 6.8 is the frequency response function form of Equation 4.9 ($s = j\,\omega$). The matrix equations in Equation 6.8 can be expanded as follows:

$$X_1(\omega) = H_{11}(\omega)\, F_1(\omega) + H_{12}(\omega)\, F_2(\omega)$$

$$X_2(\omega) = H_{21}(\omega)\, F_1(\omega) + H_{22}(\omega)\, F_2(\omega)$$

Note that $X_1(\omega)$ and $X_2(\omega)$ are the response of the system (in the frequency domain) due to the forces $F_1(\omega)$ and $F_2(\omega)$. Thus, $X_1(\omega)$ and $X_2(\omega)$ can be defined by the process of multiplication and addition of the known frequency response function measurements and forcing functions. The time domain response of the system, $x_1(t)$ and $x_2(t)$, can then be computed by taking the inverse Fourier transform of $X_1(\omega)$ and $X_2(\omega)$.

This synthesis procedure can be extended to systems with more than two degrees of freedom. In general, for an N-degree of freedom system:

$$\left\{ \begin{array}{l} X_1(\omega) \\ X_2(\omega) \\ . \\ . \\ . \\ X_N(\omega) \end{array} \right\} = \left[\begin{array}{llll} H_{11}(\omega) & H_{12}(\omega) & \ldots . & H_{1N}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) & \ldots . & H_{2N}(\omega) \\ . & . & \ldots . & . \\ . & . & \ldots . & . \\ . & . & \ldots . & . \\ H_{N1}(\omega) & H_{N2}(\omega) & \ldots . & H_{NN}(\omega) \end{array} \right] \left\{ \begin{array}{l} F_1(\omega) \\ F_2(\omega) \\ . \\ . \\ . \\ F_N(\omega) \end{array} \right\} \tag{6.9}$$

While Equation 6.9 is written in terms of the number of degrees of freedom N, this is the limiting theoretical consideration. In general, Equation 6.9 can be written in terms of any arbitrary number of inputs and responses such that [$H(s)$] is no longer a square matrix.

For instance, the time domain response $x_1(t)$ is just the inverse Fourier transform of $X_1(\omega)$ where:

$$X_1(\omega) = H_{11}\, F_1(\omega) + H_{11}\, F_2(\omega) + \ldots . \qquad + H_{1N}\, F_N(\omega)$$

Theoretically, the transfer function matrix [$H(s)$] consists of $N \times N$ transfer functions for an N-degree of freedom system. From a practical point of view, the measured portion of the transfer function matrix (frequency response function matrix) will consist of $N_o \times N_i$ frequency response functions regardless of the number of modes N.