3. MULTIPLE DEGREE OF FREEDOM SYSTEMS

3.1 Theory

Generally, most structures are more complicated than the single mass, spring, and damper system discussed in the previous section. The general case for a multiple degree of freedom system will be used to show how the frequency response functions of a structure are related to the modal vectors of that structure. Throughout the following section the following two degree of freedom system will be used to illustrate the concepts discussed.

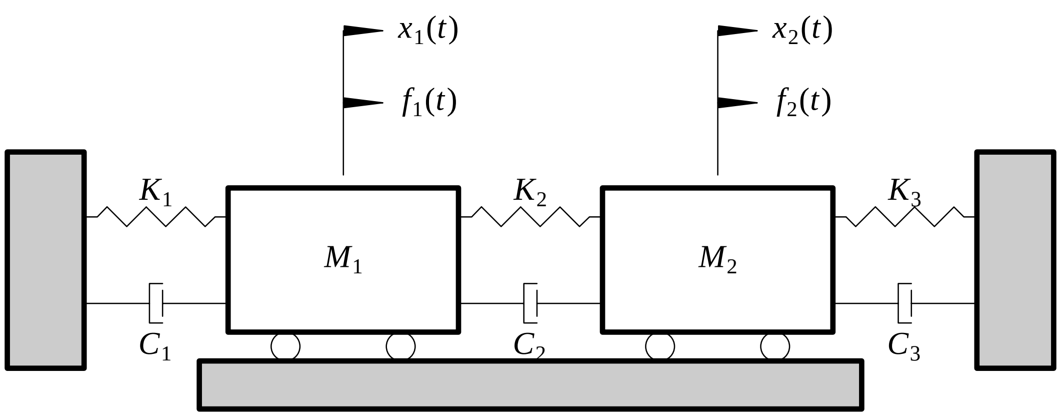

Figure 3-1. Two Degree of Freedom System

The equations of motion for the above system are:

$$M_1 \ddot{x}_1(t) + (C_1 + C_2)\ \dot{x}_1(t) - C_2\ \dot{x}_2(t) + (K_1 + K_2)\ x_1(t) - K_2\ x_2(t) = f_1\ (t)$$

$$M_2 \ddot{x}_2(t) + (C_2 + C_3)\ \dot{x}_2(t) - C_2\ \dot{x}_1(t) + (K_2 + K_3)\ x_2(t) - K_2\ x_1(t) = f_2\ (t)$$

In matrix notation:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} + \begin{bmatrix} (C_1 + C_2) & -C_2 \\ -C_2 & (C_2 + C_3) \end{bmatrix} \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix}$$

$$+ \begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & (K_2 + K_3) \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} \tag{3.1}$$

The above equations are still second order, linear, time invariant, differential equations, but are now coupled by the coordinate choice. Therefore, this system of equations must be solved simultaneously. The process of solving the set of equations in Equation 3.1 will now be reviewed in an analytical sense. The modal vectors and frequencies will result as the solution to the homogeneous portion of the differential equations summarized in Equation 3.1.

The solution of the above system of second order differential equations is first obtained for the undamped system. Assuming that $C_1 = C_2 = C_3 = 0$.

$$[\ M\ ]\{\ \ddot{x}(t)\ \} + [\ K\ ]\{\ x(t)\ \} = \{\ f(t)\ \} \tag{3.2}$$

where:

- $[\ M\ ] = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$ = Mass Matrix
- $[\ K\ ] = \begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & (K_2 + K_3) \end{bmatrix}$ = Stiffness Matrix
- $\{\ f\ \} = \begin{Bmatrix} f_1\ (t) \\ f_2\ (t) \end{Bmatrix}$ = Forcing Vector

- $\{ x \} = \left\{ \begin{matrix} x_1 (t) \\ x_2 (t) \end{matrix} \right\}$ = Response Vector

Since the forcing and response vectors are always functions of time, the functional notation (t) will be dropped in further equations.

The system of equations represented by Equation 3.2 has the general solution of:

$$\{ x \} = \{ X \} e^{s t}$$

Thus:

$$\{ \dot{x} \} = s \{ X \} e^{s t} = s \; x$$

$$\{ \ddot{x} \} = s^2 \{ X \} e^{s t} = s^2 \; x \tag{3.3}$$

where:

- $s = \sigma + j \; \omega$ = complex valued frequency

Substituting Equation 3.3 into Equation 3.2 yields:

$$s^2 [M] \{ X \} + [K] \{ X \} = \{ f \}$$

If there are no forcing function so that $\{ f \} = \{ 0 \}$, then:

$$s^2 [M] \{ X \} + [K] \{ X \} = \{ 0 \}$$

$$\left(s^2 [M] + [K] \right) \{ X \} = \{ 0 \} \tag{3.4}$$

Equation 3.4 is nothing more than a set of simultaneous algebraic equations in X_i. The unknowns are the X's and the s's. From the theory of differential equations, in order for Equation 3.4 to have other than the trivial solution, $\{ X \} = \{ 0 \}$, the determinant of the coefficients must equal zero. The determinant of the coefficients will be a polynominal in s^2. The roots of this polynominal are called eigenvalues.

In order to manipulate Equation 3.4 into a standard eigenvalue-eigenvector form, Equation 3.4 can be reformulated in a couple of different ways. First, divide Equation 3.4 by s^2 and premultiply by $[K]^{-1}$.

$$\left[[\ K\ ]^{-1} [\ M\ ] + \frac{1}{s^2} [\ I\ ] \right] \{\ X\ \} = \{\ 0\ \} \tag{3.5}$$

A different way of formulating the eigenvalue problem would be to premultiply Equation 3.4 by $[M]^{-1}$. Note that by doing this, the resulting dynamic matrix, $[\ K\ ]^{-1}[\ M\ ]$ in Equation 3.5 or $[\ M\ ]^{-1}[\ K\ ]$ in Equation 3.6, is no longer symmetric.

$$\left[[\ M\ ]^{-1} [\ K\ ] + s^2 [\ I\ ] \right] \{\ X\ \} = \{\ 0\ \} \tag{3.6}$$

In Equation 3.5 the eigenvalues are $\frac{1}{s^2}$ and in Equation 3.6 the eigenvalues are s^2. Equations 3.5 and 3.6 are really just the inverse of each other. In Equation 3.5 or Equation 3.6, the matrix on the left hand side of the equation is often referred to as the dynamic matrix. Note that the multiplication of Equation 3.4 by a matrix to obtain Equation 3.5 or 3.6 amounts to a coordinate transformation.

The frequency of a mode of vibration is defined in terms of the eigenvalue. The solution vector $\{X\}$ of Equation 3.5 or 3.6 corresponding to a particular eigenvalue is called an eigenvector, characteristic vector, mode shape, or modal vector. The X's represent a deformation pattern of the structure for a particular frequency of vibration. Since Equations 3.5 or 3.6 are homogeneous there is not a unique solution for the X's; only a relative pattern or ratio among the X's can be obtained. In other words, the X's can only be solved for in terms of one of the X's, which in turn can be given any arbitrary value. Mathematically, the rank of the equation systems represented by Equation 3.5 or 3.6 is always one less than the number of equations.

Therefore, the deflected deformation of a structure, which describes a natural mode of vibration, is defined by known ratios of the amplitude of motion at the various points on the structure. Thus, the actual amplitude of vibration of a structure is a combination of the modal vector and the level, location, and characteristic of excitation forces and not a direct property of a natural mode of vibration. The amplitude of vibration is really dependent on the placement and amplitude of the systems forcing functions along with any initial conditions of the system together with the properties of the structure described by the eigenvalues and eigenvectors.

3.2 Solution of the Eigenvalue Problem

The solution of either Equation 3.5 or 3.6 is obtained by recognizing that these equations are a set of homogeneous equations. Therefore, for a non-trivial solution, the determinant of the coefficients must equal zero.

$$\left| [\ K\ ] + s^2 [\ M\ ] \right| = 0. \tag{3.7a}$$

$$\left| [\ M\ ]^{-1} [\ K\ ] + s^2 [\ I\ ] \right| = 0. \tag{3.7b}$$

$$\left| [\ K\ ]^{-1} [\ M\ ] + \frac{1}{s^2} [\ I\ ] \right| = 0. \tag{3.7c}$$

The determinant in Equation 3.7 is referred to as the characteristic determinant. The expansion of the characteristic determinant results in the *characteristic equation* or the *frequency equation*.

Equation 3.7 may be rewritten as:

$$\alpha^n + a_1\, \alpha^{n-1} + a_2\, \alpha^{n-2} + \ldots + a_n = 0. \tag{3.8}$$

Equation 3.8 is the characteristic equation of a N-degree of freedom system, where $\alpha = s^2$ for Equation 3.7a or 3.7b or $\alpha = \frac{1}{s^2}$ for Equation 3.7c. The roots of Equation 3.8 are the eigenvalues of the system. Note that the values of s corresponding to the roots of Equation 3.8 are the complex-valued modal frequencies ($\lambda_r = \sigma_r + j\ \omega_r$).

3.2.1 Two Degree of Freedom Example: Undamped, Unforced

Given a two degree of freedom system (Equation 3.1), find its eigenvalues (undamped natural frequencies) and the respective eigenvectors (modal vectors) for the undamped system.

Referring to Figure (3-1), let:

- $M_1 = 5$ $\qquad M_2 = 10$
- $K_1 = 2$ $\qquad K_2 = 2$ $\qquad K_3 = 4$

Substituting into Equation 3.1:

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The eigenvalue problem then becomes (Equation 3.5):

$$\left[\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} + \frac{1}{s^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left[\begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} + \frac{1}{s^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or:

$$\begin{bmatrix} \frac{3}{2} + \frac{1}{s^2} & 1 \\ \frac{1}{2} & 2 + \frac{1}{s^2} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{ \ 0 \ \} \tag{3.9}$$

The determinant of the coefficient matrix of Equation 3.9 must equal zero for a non-trivial solution.

$$\left(\frac{3}{2} + \frac{1}{s^2} \right) \left(2 + \frac{1}{s^2} \right) - \frac{1}{2} = 0.$$

Using $\alpha = \frac{1}{s^2}$ as a change of variable, the characteristic equation becomes:

$$\alpha^2 + \frac{7}{2}\,\alpha + \frac{5}{2} = 0. \tag{3.10}$$

The roots of Equation 3.10 are:

$$\alpha_{1,2} = \frac{\frac{-7}{2} \pm \sqrt{49/4 - 10}}{2} = \frac{-7}{4} \pm \frac{\sqrt{9/4}}{2}$$

$$\alpha_1 = \frac{-5}{2}$$

$$\alpha_2 = -1$$

Noting the change of variable $\alpha = \frac{1}{s^2}$:

$$\alpha_1 = \frac{1}{\lambda_1^2} \qquad \alpha_2 = \frac{1}{\lambda_2^2}$$

Since $\lambda_r = \sigma_r \pm j\ \omega_r$, the complex-valued modal frequencies are:

$$\lambda_1 = \sigma_1 \pm j\ \omega_1 = \pm\ j\ \omega_1 = \pm\ j\ \sqrt{2/5}$$

$$\lambda_2 = \sigma_2 \pm j\ \omega_2 = \pm\ j\ \omega_2 = \pm\ j\ 1$$

Now the frequencies ω_1 and ω_2 can be used in Equation 3.9 to determine the modal vectors.

The modal vector for $\lambda_1 = \pm\ j\ \omega_1$ is determined using the following equations:

$$\begin{bmatrix} -1 & 1 \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{ \ 0 \ \}$$

or

$$- X_1 + X_2 = 0$$

$$X_2 = X_1$$

Thus, the modal vector corresponding to the natural frequency ω_1 is:

$$\{ \ \psi \ \}_1 = \begin{Bmatrix} X_1 \\ X_1 \end{Bmatrix}_1$$

where:

- X_1 is arbitrary (depends on scaling method)

Similarly for $\lambda_2 = \pm \ j \ \omega_2$, the modal vector is:

$$\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{ \ 0 \ \}$$

or

$$\frac{1}{2} X_1 + X_2 = 0$$

$$X_2 = \frac{-1}{2} X_1$$

or:

$$\{ \ \psi \ \}_2 = \begin{Bmatrix} X_1 \\ \frac{- X_1}{2} \end{Bmatrix}_2$$

If the deformation of $X_1 = 1$, which is an arbitrary choice depending on the scaling method, then:

For $\omega_1 = \sqrt{2/5}$:

$$\{\psi\}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}_1$$

For $\omega_2 = 1$:

$$\{\psi\}_2 = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}_2$$

3.3 Weighted Orthogonality of Modal Vectors

The solution of the eigenvalue problem as formulated in Equation 3.5 yields N natural frequencies, λ_r, and N modal vectors $\{\psi\}_r$ where N is the number of degrees of freedom of the system.

Note that any particular undamped natural frequency and the associated modal vector $\{\psi\}_r$ satisfy Equation 3.4. Thus, substituting into Equation 3.4 $s = \lambda_r$ and $\{X\} = \{\psi\}_r$ yields:

$$\lambda_r^2 [M]\{\psi\}_r = -[K]\{\psi\}_r \tag{3.11}$$

Now pre-multiply Equation 3.11 by a different modal vector, $\{\psi\}_s^T$, thus:

$$\lambda_r^2 \{\psi\}_s^T [M]\{\psi\}_r = -\{\psi\}_s^T [K]\{\psi\}_r \tag{3.12}$$

where the superscript T denotes a matrix transpose.

Using a rule of matrix algebra for the transpose of a product of matrices:

$$[[C][D]]^T = [D]^T [C]^T$$

Taking the transpose of both sides of Equation 3.12 yields:

$$\lambda_r^2 \{ \psi \}_r^T [M] \{ \psi \}_s = - \{ \psi \}_r^T [K] \{ \psi \}_s \tag{3.13}$$

where:

- $[M]^T = [M]$ since $[M]$ is a symmetric matrix.
- $[K]^T = [K]$ since $[K]$ is a symmetric matrix.

Next, substitute $s = \lambda_s$ and $\{X\} = \{\psi\}_s$ into Equation 3.4 and pre-multiply both sides by $\{\psi\}_s{}^T$. This yields:

$$\lambda_s^2 \{ \psi \}_r^T [M] \{ \psi \}_s = - \{ \psi \}_r^T [K] \{ \psi \}_s \tag{3.14}$$

Subtracting Equation 3.14 from Equation 3.13 gives:

$$(\lambda_r^2 - \lambda_s^2) \{ \psi \}_r^T [M] \{ \psi \}_s = 0. \tag{3.15}$$

If $r \neq s$ (implying two different frequencies), it follows that:

$$\{ \psi \}_r^T [M] \{ \psi \}_s = 0. \tag{3.16}$$

From Equation 3.14, it follows that:

$$\{ \psi \}_r^T [K] \{ \psi \}_s = 0. \tag{3.17}$$

Equations 3.16 and 3.17 are statements of the weighted orthogonality properties of the modal vectors with respect to the system mass and stiffness matrices. The concept of orthogonality can be looked at from a vector analysis standpoint. In vector analysis, two vectors are orthogonal if their dot product equals zero. This means that the projection of one vector on the other is zero. Therefore, the two vectors are perpendicular to each other. An obvious example is the 3-dimensional cartesian coordinate system. The i, j, and k unit vectors for the cartesian coordinate system are orthogonal to each other. Modal vectors of an n-degree of freedom system can be viewed as being just a vector in n-dimensional space, which unfortunately cannot be visualized. In order for modal vectors to be orthogonal, though, a simple dot product will not suffice. The concept of a weighted dot product, where the weighting matrix is the theoretical mass or stiffness matrix, must be used. If, for instance, the mass matrix in Equation 3.16 was the identity matrix, the weighted dot product would reduce to the simple dot product and result in a direct analog of the orthogonality condition for the unit vectors in the cartesian coordinate system. Because the mass and stiffness matrices in Equation 3.16 and 3.17 are not generally the identity matrix, the orthogonality relationships in Equation 3.16 and Equation 3.17 are generally referred to as weighted orthogonality.

If two modal vectors happen to have the same frequency $\lambda_r = \lambda_s$ (Equation 3.15) their corresponding modal vectors are not necessarily orthogonal to one another. This condition is known as a *repeated root* or *repeated pole* and will be discussed further in a later section. For this condition, the modal vectors associated with the repeated roots will be orthogonal to the other modal vectors and independent of one another.

In Equation 3.15, if the same modal vector is used to pre- and post-multiply the mass matrix, then Equation 3.16 is equal to some scalar constant other than zero, commonly noted as M_r. Thus:

$$\{ \psi \}_r^T [M] \{ \psi \}_r = M_r = \text{Modal Mass} \tag{3.18}$$

Similarly, Equation 3.14 yields:

$$\{ \psi \}_r^T [K] \{ \psi \}_r = \omega_r^2 M_r = K_r = \text{Modal Stiffness} \tag{3.19}$$

Since, as previously shown, the amplitude of any particular modal vector (eigenvector) is completely arbitrary, the modal vector can be normalized in an arbitrary way. This means that M_r is not unique.

For instance, one common criteria used to normalize the modal vector is to scale the modal vector such that M_r in Equation 3.18 is equal to unity.

The resulting scaled modal vectors normalized in this manner are generally referred to as orthonormal modal vectors (eigenvectors).

3.4 Modal Vector Scaling Example

Using the previous two degree of freedom example, normalize the modal vectors $\{\psi\}_1$ and $\{\psi\}_2$ such that:

$$\{\psi\}_1^T [M] \{\psi\}_1 = M_1 = 1. \tag{3.20}$$

and:

$$\{\psi\}_2^T [M] \{\psi\}_2 = M_2 = 1. \tag{3.21}$$

From the previous example:

$$\{\psi\}_1 = \begin{Bmatrix} X_1 \\ X_1 \end{Bmatrix}_1 \qquad \{\psi\}_2 = \begin{Bmatrix} X_1 \\ \dfrac{-X_1}{2} \end{Bmatrix}_2$$

Substituting $\{\psi\}_1$ into Equation 3.20 yields:

$$\begin{Bmatrix} X_1 \\ X_1 \end{Bmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_1 \end{Bmatrix} = 1$$

$$\begin{Bmatrix} 5\, X_1 \\ 10\, X_1 \end{Bmatrix}^T \begin{Bmatrix} X_1 \\ X_1 \end{Bmatrix} = 1$$

$$5\, X_1^2 + 10\, X_1^2 = 1$$

$$X_1^2 = \frac{1}{15}$$

$$X_1 = \pm \sqrt{1/15}$$

Using the positive root, the modal vector $\{\psi\}_1$, normalized for unity modal mass, results:

$$\{\psi\}_1 = \begin{Bmatrix} X_1 \\ X_1 \end{Bmatrix}_1 = \begin{Bmatrix} \sqrt{1/15} \\ \sqrt{1/15} \end{Bmatrix}_1$$

Similarly for $\{\psi\}_2$:

$$\left\{ \begin{array}{c} X_1 \\ \dfrac{-X_1}{2} \end{array} \right\}^T \left[\begin{array}{cc} 5 & 0 \\ 0 & 10 \end{array} \right] \left\{ \begin{array}{c} X_1 \\ \dfrac{-X_1}{2} \end{array} \right\} = 1$$

$$\left\{ \begin{array}{c} 5\ X_1 \\ -5\ X_1 \end{array} \right\}^T \left\{ \begin{array}{c} X_1 \\ \dfrac{-X_1}{2} \end{array} \right\} = 1$$

$$5\ X_1^2 + \frac{5}{2}\ X_1^2 = 1$$

$$X_1^2 = \frac{2}{15}$$

$$X_1 = \pm \sqrt{2/15}$$

Thus, $\{\psi\}_2$ normalized to unity modal mass is:

$$\{\ \psi\ \}_2 = \left\{ \begin{array}{c} \sqrt{2/15} \\ \dfrac{-\sqrt{2/15}}{2} \end{array} \right\}_2$$

The normalized modal vectors will give $M_r = 1$ for all the modes of vibration. The significance of this normalization will be obvious later.

3.5 Principal Coordinates - Modal Coordinates

With reference to the equations of motion for an undamped system (Equation 3.2), the major obstacle encountered when trying to solve for the system response $\{x\}$, due to a particular set of forcing functions and initial conditions, is the coupling between the equations. In terms of the system's mass and stiffness matrices, coupling is represented in terms of non-zero off diagonal elements. Generally two types of coupling can exist for an undamped system; (1) Static coupling (non-diagonal stiffness matrix); or (2) dynamic coupling (non-diagonal mass matrix). Equation 3.2 represents a system which is only statically coupled. If the system of equations in Equation 3.2 could be uncoupled, that is diagonal mass and stiffness matrices, then each equation in Equation 3.2 could be solved independent of the other equations. Another way of looking at this would be that each uncoupled equation would look just like the equation for a single degree of freedom, whose solution can very easily be obtained. Therefore, if a set of coupled system equations could be reduced to an uncoupled system, the solution would become straightforward. Indeed, from an analytical sense, this is the whole point of what has become known as modal analysis.

The procedure used to uncouple a set of coupled system equations is basically a coordinate transformation. In other words, the goal is to find a coordinate transformation that transforms the original coordinates $\{x\}$ into another equivalent set of coordinates $\{q\}$ that renders the system statically and inertially decoupled. This new set of coordinates $\{q\}$ is typically referred to as principal coordinates, normal coordinates or modal coordinates.

A similar benefit of a coordinate transformation occurs in many other engineering problems. One example of this situation is in the calculation of moments and products of inertia when the inertia properties of a complex structure need to be defined. The first step in the calculation of the inertia properties is to choose a set of axis to base the inertia properties on. Then, the following properties would be measured or calculated: I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{xz}. In general, both moments of inertia and products of inertia are required. However, if a different set of axis with respect to the structure were defined such that these axis happened to coincide with the structures principle axis, the result would be moments of inertia I_x, I_y, and I_z but the products of inertia would all be zero ($I_{xy} = I_{yz} = I_{xz} = 0$). Therefore, by changing the coordinate system, the products of inertia have been eliminated.

Another example of the benefit of a coordinate transformation is noticed when computing principle strains at a point on a structure. Typically, a strain gage rosette is used to determine the normal and shearing strains at a point of interest. From this information, a new coordinate system can be determined (strain element orientation) such that only principal normal strains exist; the shear strains are equal to zero for the new coordinate system. To determine the orientation of this new coordinate system that renders the shearing strain to zero, MOHR's circle techniques are commonly used. Once again, a simple coordinate transformation is used to eliminate the shearing strains.

The problem of finding a coordinate transformation that uncouples our original equations of motion is very straightforward. It turns out that, due to the unique orthogonality properties of the modal vectors, the required coordinate transformation is already available. Referring to Equations 3.16-3.19, if either the mass or stiffness matrix is pre- and post- multiplied by different modal vectors (Equations 3.16 and 3.17), the result is zero. However, if the same modal vector is used to pre- and post- multiply the mass or stiffness matrix (Equations 3.18 and 3.19), the result is a constant.

Therefore, the new coordinate system can be defined by the following transformation:

$$\{ x \} = \left[\psi \right] \{ q \} \tag{3.22}$$

where:

- $[\psi]$ is the transformation matrix (matrix whose columns are the modal vectors of the original system).

This matrix is generally referred to as the *modal matrix* or *matrix of modal vectors*. Recall the general form of the undamped system of equations with forcing functions:

$$[M] \{ \ddot{x} \} + [K] \{ x \} = \{ f \} \tag{3.23}$$

Substituting Equation 3.22 into Equation 3.23 gives:

$$[M] \left[\psi \right] \{ \ddot{q} \} + [K] \left[\psi \right] \{ q \} = \{ f \} \tag{3.24}$$

Pre-multiplying by $\left[\psi \right]^T$ yields:

$$\left[\psi \right]^T [M] \left[\psi \right] \{ \ddot{q} \} + \left[\psi \right]^T [K] \left[\psi \right] \{ q \} = \left[\psi \right]^T \{ f \}$$

Equation 3.25 is the equivalent of Equation 3.23 but in a different coordinate system. Analyzing

Equation 3.25, noting the orthogonality properties of the modal vectors (Equation 3.16-3.19):

$$\begin{bmatrix} \psi \end{bmatrix}^T [\, M \,] \begin{bmatrix} \psi \end{bmatrix} = \lceil M \rfloor$$

and:

$$\begin{bmatrix} \psi \end{bmatrix}^T [\, K \,] \begin{bmatrix} \psi \end{bmatrix} = \lceil K \rfloor$$

where:

- $\lceil M \rfloor$ is a diagonal matrix.
- $\lceil K \rfloor$ is a diagonal matrix.

Therefore, Equation 3.25 becomes:

$$\lceil M \rfloor \{\, \ddot{q} \,\} + \lceil K \rfloor \{\, q \,\} = \begin{bmatrix} \psi \end{bmatrix}^T \{\, f \,\} \tag{3.26}$$

From inspection, since both the new mass and stiffness matrices are diagonal, the coordinate transformation $\{x\} = [\psi]\, \{q\}$ has completely uncoupled the set of equations. Now each equation in Equation 3.26 is an equation for a single degree of freedom oscillator which is easily solved.

The r-th Equation of Equation 3.26 is:

$$M_r\, \ddot{q}_r + K_r\, q_r = \{\, \psi \,\}_r^T \{\, f \,\} = f_r \tag{3.27}$$

This is the equation of motion for the single degree of freedom system shown below.

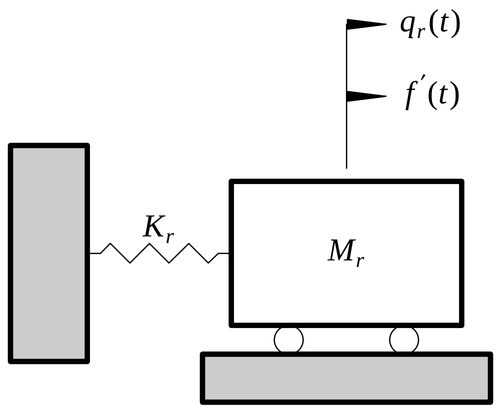

Figure 3-2. Single Degree of Freedom System

The quantity M_r is called the modal mass or generalized mass for the r-th mode of vibration. The quantity K_r is called the modal stiffness or generalized stiffness for the r-th modal vector of vibration. While these quantities are viewed as mass and stiffness related, it is important to remember that the magnitude of these quantities depends upon the scaling of the modal vectors. Therefore, although both the modal vectors and the modal mass/stiffness quantities are computed in a relative manner, only the combination of a modal vector together with the associated modal mass represents a unique absolute characteristic concerning the system being described.

It has been shown previously that the modal vectors may be normalized such that $M_r = 1$. If this has been done, then Equation 3.27 can be rewritten for the r-th modal vector as:

$$\ddot{q}_r + \Omega_r^2 \, q_r = f^{\,\prime} \tag{3.28}$$

where:

- $M_r = 1.0$
- $K_r = {\Omega_r}^2$

Once the solution (time responses) of Equation 3.26 for all q's has been computed, the solution in terms of the original coordinates can then be obtained through the use of the coordinate transformation equation in Equation 3.22.

3.6 Two Degree of Freedom Example: Undamped, Forced

Referring to the previous example, some forcing functions can now be included in the system of equations.

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \{ \ddot{x} \} + \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \{ x \} = \{ f \} \tag{3.29}$$

The natural frequencies and normalized modal vectors of the above system are:

For $\omega_1 = \sqrt{2/5}$:

$$\{ \psi \}_1 = \begin{Bmatrix} \sqrt{1/15} \\ \sqrt{1/15} \end{Bmatrix}_1$$

For $\omega_2 = 1$:

$$\{ \psi \}_2 = \begin{Bmatrix} \sqrt{2/15} \\ -\dfrac{\sqrt{2/15}}{2} \end{Bmatrix}_2$$

Forming the modal matrix:

$$\begin{bmatrix} \psi \end{bmatrix} = \begin{bmatrix} \{ \psi \}_1 & \{ \psi \}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{1/15} & \sqrt{2/15} \\ \sqrt{1/15} & -\dfrac{\sqrt{2/15}}{2} \end{bmatrix}$$

Now make the following coordinate transformation:

$$\{ x \} = \begin{bmatrix} \psi \end{bmatrix} \{ q \} \tag{3.30}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \sqrt{1/15} & \sqrt{2/15} \\ \sqrt{1/15} & -\dfrac{\sqrt{2/15}}{2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Substituting Equation 3.30 into Equation 3.29 and pre-multiplying by $\begin{bmatrix} \psi \end{bmatrix}^T$ yields:

$$\begin{bmatrix} \psi \end{bmatrix}^T [M] \begin{bmatrix} \psi \end{bmatrix} \{ q \} + \begin{bmatrix} \psi \end{bmatrix}^T [K] \begin{bmatrix} \psi \end{bmatrix} \{ q \} = \begin{bmatrix} \psi \end{bmatrix}^T \{ f(t) \}$$

$$\begin{bmatrix} \psi \end{bmatrix}^T [M] \begin{bmatrix} \psi \end{bmatrix} = \begin{bmatrix} \sqrt{1/15} & \sqrt{1/15} \\ \sqrt{2/15} & -\dfrac{\sqrt{2/15}}{2} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} \sqrt{1/15} & \sqrt{2/15} \\ \sqrt{1/15} & -\dfrac{\sqrt{2/15}}{2} \end{bmatrix}$$

$$\begin{bmatrix} \psi \end{bmatrix}^T [M] \begin{bmatrix} \psi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\ \psi\ \right]^T\left[\ K\ \right]\left[\ \psi\ \right]=\begin{bmatrix}\sqrt{1/15} & \sqrt{1/15}\\ \sqrt{2/15} & -\dfrac{\sqrt{2/15}}{2}\end{bmatrix}\begin{bmatrix}4 & -2\\ -2 & 6\end{bmatrix}\begin{bmatrix}\sqrt{1/15} & \sqrt{2/15}\\ \sqrt{1/15} & -\dfrac{\sqrt{2/15}}{2}\end{bmatrix}$$

$$\left[\ \psi\ \right]^T\left[\ K\ \right]\left[\ \psi\ \right]=\begin{bmatrix}2/5 & 0\\ 0 & 1\end{bmatrix}$$

Therefore, the new equations of motion are:

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix}+\begin{bmatrix}2/5 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}=\begin{Bmatrix}\sqrt{1/15}\,f_1+\sqrt{1/15}\,f_2\\ \sqrt{2/15}\,f_1-\dfrac{\sqrt{2/15}}{2}\,f_2\end{Bmatrix}=\begin{Bmatrix}f_1^{\prime}\\ f_2^{\prime}\end{Bmatrix}$$

The matrix equation of Equation 3.31 can now be written in terms of algebraic differential equations:

$$\ddot{q}_1+\frac{2}{5}\,q_1=f_1^{\prime} \tag{3.32}$$

$$\ddot{q}_2+q_2=f_2^{\prime} \tag{3.33}$$

Hence, the system equations have been uncoupled by using the modal matrix as a coordinate transformation.

The original system looked like:

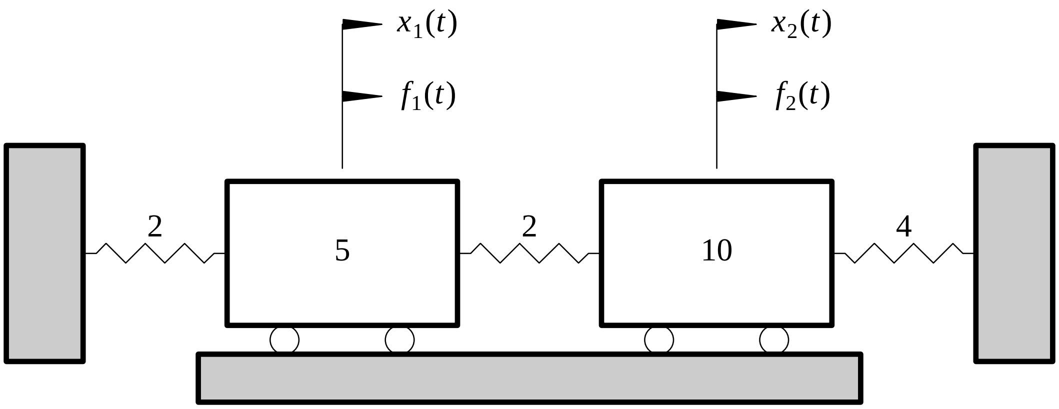

Figure 3-3. Original System

The transformed system can be pictured as:

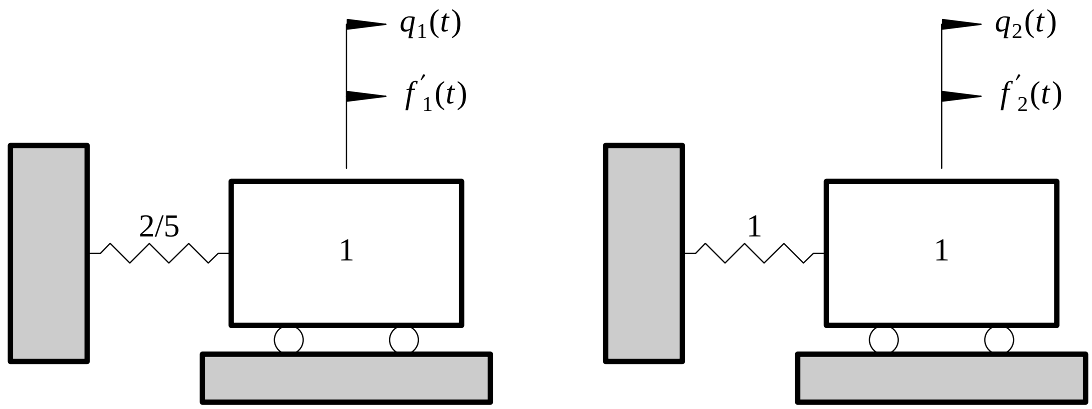

Figure 3-4. Transformed System

Once $q_1(t)$ and $q_2(t)$ are known, Equation 3.30 can be used to compute $x_1(t)$ and $x_2(t)$. Thus:

$$x_1(t) = \sqrt{1/15}\, q_1(t) + \sqrt{2/15}\, q_2(t)$$

$$x_2(t) = \sqrt{1/15}\, q_1(t) - \frac{\sqrt{2/15}}{2}\, q_2(t)$$

Many points should be emphasized from the previous discussion. Modal vectors, along with their frequencies, are a dynamic property of a structure. The amplitudes of a modal vector are completely arbitrary; that is, only the ratios between the components of a particular modal vector are unique. Because of the orthogonality properties of the modal vectors, with respect to the system's mass and stiffness matrices, modal mass and modal stiffness can be defined. These quantities depend upon the scaling of the modal vectors, so that the absolute magnitudes of these quantities are also arbitrary. Finally, a simple coordinate transformation (modal matrix) can be used to represent a complicated interconnection of springs and masses as a collection of single degree of freedom oscillators.

3.7 Proportional Damping

In order to evaluate multiple degree of freedom systems that are present in the real world, the effect of damping on the complex frequencies and modal vectors must be considered. Many physical mechanisms are needed to describe all of the possible forms of damping that may be present in a particular structure or system. Some of the classical types are:

- Structural Damping
- Viscous Damping
- Coulomb Damping
- Hysteretic Damping

It is generally difficult to ascertain which type of damping is present in any particular structure. Indeed most structures exhibit damping characteristics that result from a combination of all the above, plus others that have not been described here.

It will suffice to say that whenever a structure is modeled with a particular form of damping, for example, viscous, that the damping model is an equivalent model to whatever type of damping that may actually be present.

Rather than consider the many, different physical mechanisms, the probable location of each mechanism, and the particular mathematical representation of the mechanism of damping that is needed to describe the dissipative energy of the system, a model will be used that is only concerned with the resultant mathematical form. This model will represent a hypothetical form of damping, that is proportional to the system mass or stiffness matrix. Therefore:

$$[\, C \,] = \alpha \, [\, M \,]$$

or :

$$[\, C \,] = \beta \, [\, K \,]$$

The most common formulation for proportional damping is:

$$[\ C\ ] = \alpha\,[\ M\ ] + \beta\,[\ K\ ]$$

where:

- $[\ C\ ]$ = damping matrix
- α , β = constants

Note that the case of no damping is the trivial proportional damped case with both coefficients equal to zero. While the above definition is sufficient for most cases, the theoretical relationship between mass, stiffness and damping matrices can be somewhat more complicated and still qualify as proportional damping. Theoretically, any damping matrix that satisfies the following relationship will yield proportional damping with all the qualifications (normal modes) involved in subsequent discussion.

$$\Big[[M]^{-1}[C]\Big]^s\Big[[M]^{-1}[K]\Big]^r = \Big[[M]^{-1}[K]\Big]^r\Big[[M]^{-1}[C]\Big]^s$$

where:

- r and s = integers.

For the purposes of most practical problems, the simpler relationship will be sufficient.

3.8 Modal Vectors from the System Matrix

The modal vectors can be determined in a somewhat more direct manner through a manipulation of the system matrix. Understanding this approach to the evaluation of modal vectors is very useful in relating measured frequency response function data to the system modal vectors.

Starting with Equation 3.4:

$$\Big[[\ M\ ]s^2 + [\ C\ ]s + [\ K\ ]\Big]\{\ X\ \} = \{\ 0\ \} \qquad (3.34)$$

Define:

$$[\ B(s)\ ] = \Big[[\ M\ ]s^2 + [\ C\ ]s + [\ K\ ] \Big]$$

where:

- $[\ B(S)\ ]$ = System Impedance Matrix

From matrix algebra:

$$[\ B(s)\ ][\ B(s)\ ]^{-1} = [\ I\ ] \tag{3.35}$$

$$[\ B(s)\ ]^{-1} = \frac{[\ B(s)\ ]^A}{|\ [\ B(s)\ ]\ |} \tag{3.36}$$

where:

- $[\ B(s)\ ]^A$ is the adjoint of matrix $[\ B(s)\ ]$.

Substituting Equation 3.36 into Equation 3.35 yields:

$$[\ B(s)\ ][\ B(s)\ ]^A = |[\ B(s)\ ]|\ [\ I\ ] \tag{3.37}$$

If λ_r is a root of the characteristic equation from Equation 3.34, then $|\ [\ B(\lambda_r)\ ]\ | = 0$.

Evaluating Equation 3.37 at $s = \lambda_r$ gives:

$$\Big[\ B(\lambda_r)\ \Big]\Big[\ B(\lambda_r)\ \Big]^A = [\ 0\ ] \tag{3.38}$$

Equation 3.38 can be rewritten using any column of $[\ B(\lambda_r)\ ]^A$, the $i-th$ column for example $\{\ B(\lambda_r)\ \}_i^A$. Therefore:

$$\Big[\ B(\lambda_r)\ \Big]\{\ B(\lambda_r)\ \}_i^A = \{\ 0\ \} \tag{3.39}$$

Equation 3.39 represents a set of homogeneous equations in $\{ B(\lambda_r) \}_i^A$ which determines each element of $\{ B(\lambda_r) \}_i^A$ to within an arbitrary constant. Note that the constant will be different depending upon the column that is used.

Evaluating Equation 3.34 at one of the eigenvalues of the system (λ_r):

$$\left[B(\lambda_r) \right] \{ X \}_r = \{ 0 \} \tag{3.40}$$

Equation 3.40 (formerly Equation 3.34), just like Equation 3.39, represents a set of homogeneous equations in $\{ X \}$ Equation 3.34 is evaluated at a specific eigenvalue, the resulting solution is the eigenvector coresponding to the specific eigenvalue. This eigenvector is determined to within an arbitrary constant. Therefore, from Equation 3.39 and Equation 3.40, $\{ B(\lambda_r) \}_i^A$ and $\{X\}_r$ are proportional and both represent the eigenvector corresponding to the eigenvalue λ_r. Recall that $\{X\}_r$ (Equation 3.40) has been previously shown to be the r-th modal vector of the system. Therefore:

$$\{ X \}_r = \beta_{ir} \{ B(\lambda_r) \}_i^A$$

where:

- β_{ir} is a proportionality constant.

Note: One of the major points is that the columns of the adjoint matrix $[B(\lambda_r)]^A$ are all proportional to the r-th modal vector.

Since the mass, damping and stiffness matrices are assumed to be symmetric when absolute coordinates are used (and proportional damping is present), the system impedance matrix $[B(s)]$ is symmetric. Therefore, the adjoint matrix of $[B(\lambda_r)]$ is also symmetric. Thus, the rows of the adjoint matrix are also proportional to the modal vector. The adjoint matrix can now be expressed for the r-th mode in terms of the r-th modal vector.

$$\left[B(\lambda_r) \right]^A = \gamma_r \{ \psi \}_r \{ \psi \}_r^T \tag{3.41}$$

$$\left[\; B \,(\, \lambda_r \,) \; \right]^A = \gamma_r \begin{bmatrix} \psi_1 \, \psi_1 & \psi_1 \, \psi_2 & \cdot & \cdot & \cdot & \cdot & \psi_1 \, \psi_N \\ \psi_2 \, \psi_1 & \psi_2 \, \psi_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \psi_N \, \psi_1 & \psi_N \, \psi_2 & \cdot & \cdot & \cdot & \cdot & \psi_N \, \psi_N \end{bmatrix}_r$$

where:

- $\gamma_r =$ constant associated with the scaling of $\{\psi\}_r$ relative to the absolute scaling (units) of the adjoint matrix.

Note that the adjoint matrix is not the same as the modal matrix since each column of the adjoint matrix is proportional to the same modal vector. Therefore, the adjoint matrix needs to be evaluated for each of the N eigenvalues to determine the N eigenvectors. Also note that, due to the symmetry of the adjoint matrix, if one of the modal coefficients is zero, then a complete row and column of the corresponding adjoint matrix will be zero. This phenomenon is normal and corresponds to physically trying to excite (force) the system at the node (modal coefficient equal to zero) of one of the modal vectors of the system. Theoretically, the corresponding mode of vibration will not be observed in this situation. Analytically, this problem can be overcome by evaluating a different row or column of the adjoint matrix. Experimentally, the configuration of the input and/or output sensors may have to be altered to detect this case.

Equation 3.41 is extremely important and will be used in the next section to show that the residues of a frequency response function for a particular pole (λ_r) are directly related to the elements of a modal vector. Also, the symmetry of the adjoint matrix is the justification for not needing to evaluate the complete frequency response function matrix.