7. MODAL SCALING

7.1 Proportionally Damped Systems (Modal Mass)

The modal matrix (matrix of modal vectors) has been previously used as a coordinate transformation in order to diagonalize the mass, damping (if proportional), and stiffness matrices. The diagonalization of these matrices leads to the analytical definition of modal mass, modal damping, and modal stiffness. From an experimental view point, the mass, damping, and stiffness matrices are generally not known. Therefore, this theoretical approach to the definition of modal mass, modal damping, and modal stiffness is not useful. Even so, the modal mass, modal damping, and modal stiffness can be computed directly from the measured frequency response functions without the benefit of prior knowledge of the mass, damping, and stiffness matrices. Note that the modal mass, modal damping, and modal stiffness are, in general, not physical properties but are generalized, or normalized, properties related to the physical properties.

Recall that, analytically, the value of the modal mass is completely dependent on the scaling chosen for the modal vectors. Any development from the frequency response function (experimental) must be consistent with this concept. The development in Section 4.7 (Equation 4.51) satisfies this constraint. Equation 4.51 is repeated here as Equation 7.1.

$$M_r = \frac{1}{j\,2\,Q_r\,\omega_r} \tag{7.1}$$

The scaling coefficient Q_r in the above definition can only be found after the choice of modal vector scaling is made.

Recal that for the r-th mode of an N degree of freedom system:

$$[\ A\ ]_r = Q_r\ \{\ \psi\ \}_r\ \{\ \psi\ \}_r^T \tag{7.2}$$

$$[\ A\ ]_r = Q_r \begin{bmatrix} \psi_1\psi_1 & \psi_1\psi_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \psi_1\psi_m \\ \psi_2\psi_1 & \psi_2\psi_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \psi_m\psi_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \psi_m\psi_m \end{bmatrix}_r \tag{7.3}$$

where:

- Q_r is the scaling constant that is a function of the scaling of the modal vectors.

Using only the q-th column of the residue matrix:

$$\begin{bmatrix} A_{1q} \\ A_{2q} \\ \cdot \\ \cdot \\ A_{iq} \\ \cdot \\ \cdot \\ A_{mq} \end{bmatrix}_r = Q_r \begin{bmatrix} \psi_1 \psi_q \\ \psi_2 \psi_q \\ \cdot \\ \cdot \\ \psi_i \psi_q \\ \cdot \\ \cdot \\ \psi_m \psi_q \end{bmatrix}_r = Q_r \ \psi_{qr} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_i \\ \cdot \\ \cdot \\ \psi_m \end{bmatrix}_r \tag{7.4}$$

The relationship between the residue, the scaling constant and the scaling of the modal vector is most clear when the FRF measurement $H_{qq}(\omega)$ is utilized. Note that $H_{qq}(\omega)$ is obtained by exciting at point q and measuring the response at point q. This is typically referred to as the *driving point* frequency response function. Therefore, the A_{qqr} residue for all modes $r = 1 \rightarrow N$ can be determined from the $H_{qq}(\omega)$ measurement.

Now, note the q-th element of Equation 7.4:

$$A_{qqr} = Q_r \ \psi_{qr} \ \psi_{qr} = Q_r \ \psi_{qr}^2 \tag{7.5}$$

Therefore, the r-th modal mass of a multi-degree of freedom system is defined as:

Modal Mass

$$M_r = \frac{1}{j \ 2 \ Q_r \ \omega_r} \tag{7.6}$$

In general:

$$M_r = \frac{\psi_{pr} \ \psi_{qr}}{j \ 2 \ A_{pqr} \ \omega_r} \tag{7.7}$$

where:

- M_r = Modal mass
- Q_r = Modal scaling constant
- ω_r = Damped natural frequency

While this concept of modal mass being a relative quantity (relative to scaling) is at odds with the engineering view of mass being an absolute quantity, this is consistent with the analytical definition of modal mass.

While many choices of modal vector scaling exist, choosing to scale the modal vector such that the largest modal coefficient will be equal to 1.0 gives a result where the modal mass will always be bounded between zero and the physical mass of the system. For the case where the modal vector of the system describes the rigid body translation (bounce) of the system, this particular choice will give the mass of the system as the modal mass. *Therefore, if the largest scaled modal coefficient is equal to unity, Equation 7.7 will compute a quantity of modal mass that has physical significance. The physical significance is that the quantity of modal mass computed under these conditions will be a number between zero and the total mass of the system.* Therefore, under this scaling condition, the modal mass can be viewed as the amount of mass that is participating in each mode of vibration.

Note that the *modal mass* defined in Equation 7.6 or 7.7 is developed in terms of displacement over force units. If measurements, and therefore residues are developed in terms of any other units (velocity over force or acceleration over force), either the measurements or Equation 7.6 and 7.7 will have to be altered accordingly.

Once the modal mass is known, the modal damping and stiffness can be obtained through the following SDOF relationships:

Modal Damping

$$C_r = 2\ \sigma_r\ M_r \tag{7.8}$$

Modal Stiffness

$$K_r = (\ \sigma_r^2 + \omega_r^2\ )\ M_r = \Omega_r^2\ M_r \tag{7.9}$$

7.1.1 Modal Vector Scaling

There are many ways to scale the modal vectors, however the following approaches are typically used:

- Unity Modal Mass
- Unity Modal Coefficient
- Unity Modal Vector Length

The scaling value, Q_r, can be calculated for each of the three cases.

7.1.1.1 Unity Modal Mass

$$Q_r = \frac{1}{j\ 2\ M_r\ \omega_r} = \frac{1}{j\ 2\ \omega_r}$$

The scaled modal coefficient at the driving point can now be computed as follows:

$$Q_r\ \psi_{qr}\ \psi_{qr} = A_{qqr}$$

Dividing both sides by Q_r:

$$\psi_{qr}\ \psi_{qr} = \frac{A_{qqr}}{Q_r}$$

The scaled modal coefficient at the driving point can now be found. For the proportionally damped case, this is obviously trivial since the square root of both sides of the above equation involves the square root of a real-valued number. For the general case, the modal coefficient can be complex, and the above equation must be solved for the complex modal coefficient. In this case the square root cannot be used.

Thus, the scaled modal vector is:

$$\{ \psi \}_r = \frac{1}{Q_r \ \psi_{qr}} \{ A \}_r$$

7.1.1.2 Unity Modal Coefficient

Assume that the i-th component ψ_i of a modal vector must be set equal to unity ($\psi_i = 1.0$). Then:

$$\{ \psi \}_r = \frac{1}{A_{iq}} \{ A \}_r$$

Thus:

$$Q_r = \frac{A_{iqr}}{\psi_{ir} \ \psi_{qr}}$$

7.1.1.3 Unity Modal Vector Length

$$\{ \psi \}_r = \frac{1}{\|\{A\}_r\|_2} \{ A \}_r$$

where:

- $\|\{A\}_r\|_2 \ = $ vector norm of $\{A\}_r$
- $\|\{A\}_r\|_2 \ = \sqrt{\sum_{i=1}^{m} A_{iqr} \ A^*_{iqr}}$

Thus:

$$Q_r = \frac{A_{iqr}}{\psi_{ir} \ \psi_{qr}}$$

7.1.2 Modal Vector Scaling Example

Using the modal vectors from the example in Chapter 5, the modal vector scaling required for unity modal mass can be determined.

The modal vectors for modes 1 and 2 are:

$$\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}_1 = \begin{bmatrix} -j \dfrac{\sqrt{39}}{117} \\ -j \dfrac{\sqrt{39}}{117} \end{bmatrix} \qquad \lambda_1 = -\frac{1}{10} + j\,\frac{\sqrt{39}}{10}$$

$$\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}_2 = \begin{bmatrix} -j \dfrac{4\,\sqrt{15}}{225} \\ +j \dfrac{2\,\sqrt{15}}{225} \end{bmatrix} \qquad \lambda_2 = -\frac{1}{4} + j\,\frac{\sqrt{15}}{4}$$

For unity modal mass, the scaling factor can be evaluated from Equation 7.6.

$$Q_r = \frac{1}{j\,2\,\omega_r}$$

Thus, the scaled modal vector for mode 1 for a modal mass of unity is:

$$\begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix}_1 = Q_1\,\psi_1 \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 \qquad \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 = \begin{Bmatrix} \sqrt{1/15} \\ \sqrt{1/15} \end{Bmatrix}_1$$

The scaled modal vector for mode 2 for a modal mass of unity is:

$$\begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix}_2 = Q_2\,\psi_1 \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_2 \qquad \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_2 = \begin{Bmatrix} \sqrt{2/15} \\ -\dfrac{1}{2}\,\sqrt{2/15} \end{Bmatrix}_2$$

Notice that these modal vectors that have been scaled with respect to unity modal mass, are identical to the modal vectors previously computed (Section 3.2). The important concept, though, is that these scaled modal vectors were determined directly from the frequency response functions. In other words, the mass, damping, and stiffness of the system did not have to be known in order to determine the same information.

The modal damping and modal stiffnesses can now be calculated using Equations 7.8.and 7.9.

$$C_1 = 2\ \sigma_1\ M_1 \qquad = 2\ (\frac{1}{10})\ 1 \qquad = \frac{1}{5}$$

$$C_2 = 2\ \sigma_2\ M_2 \qquad = 2\ (\frac{1}{4})\ 1 \qquad = \frac{1}{5}$$

$$K_1 = (\sigma_1^2 + \omega_1^2\ )\ M_1 \qquad = (\frac{1}{100} + \frac{39}{100})\ 1 \qquad = \frac{2}{5}$$

$$K_2 = (\sigma_2^2 + \omega_2^2\ )\ M_2 \qquad = (\frac{1}{16} + \frac{15}{16})\ 1 \qquad = 1$$

These are exactly the same parameters as those calculated previously.

7.2 Non-Proportionally Damped Systems (Modal A)

Consistent with the definition of *modal A* as the modal scaling factor used for the theoretical case of nonproportionally damped systems, the *modal A* scaling factor is also the basis for the relationship between the scaled modal vectors and the residues determined from the measured frequency response functions. In general, for most experimental work, *modal A* is used as the default scaling approach. If modal mass needs to be estimated, under the contraint of real (normal) modes, modal mass can be estimated from *modal A.* The following development explains the relationship between *modal A* and modal mass. Starting with the anlytical definition of *modal mass* in 2N space:

$$M_{A_r} = \{\phi\}_r^T \begin{bmatrix} [\,0\,] & [\,M\,] \\ [\,M\,] & [\,C\,] \end{bmatrix} \{\phi\}_r \tag{7.10}$$

$$M_{A_r} = \begin{Bmatrix} \lambda_r \{\psi\}_r \\ \{\psi\}_r \end{Bmatrix}^T \begin{bmatrix} [\,0\,] & [\,M\,] \\ [\,M\,] & [\,C\,] \end{bmatrix} \begin{Bmatrix} \lambda_r \{\psi\}_r \\ \{\psi\}_r \end{Bmatrix} \tag{7.11}$$

Multiplying Equation 7.11 out in terms of N space yields:

$$M_{A_r} = \lambda_r \, \{\psi\}_r^T \, [\, M \,] \, \{\psi\}_r + \lambda_r \, \{\psi\}_r^T \, [\, M \,] \, \{\psi\}_r + \{\psi\}_r^T \, [\, C \,] \, \{\psi\}_r \tag{7.12}$$

If the system is proportionally damped, the weighted orthogonality relationships between the mass matrix and the modal vectors can now be applied.

$$M_{A_r} = \lambda_r \, M_r + \lambda_r \, M_r + C_r = 2 \, \lambda_r \, M_r + C_r \tag{7.13}$$

Applying the SDOF relationship between the modal damping and modal mass ($C_r = -\sigma_r \, M_r$):

$$M_{A_r} = 2 \, \lambda_r \, M_r - 2\sigma_r \, M_r \tag{7.14}$$

$$M_{A_r} = 2 \, (\sigma_r + j \, \omega_r) \, M_r - 2\sigma_r \, M_r \tag{7.15}$$

$$M_{A_r} = j2\omega_r \, M_r \tag{7.16}$$

Equation 7.16 indicates that, for proportionally damped systems, if the modal vectors are scaled to give real valued normal modes and the modal mass is, therefore, real valued, the associated *modal A* for a proportionally damped system is imaginary valued.

While Equation 7.16 is valid only for proportionally damped systems, another form of Equation 7.16 gives a more general result that includes any type of damping. Equation 4.51 is repeated here for convenience.

$$M_r = \frac{1}{j2\omega_r Q_r} \tag{7.17}$$

Plugging Equation 7.17 into 7.16 yields:

$$M_{A_r} = \frac{1}{Q_r} \tag{7.18}$$

Since the basic development of the relationship between the residue and the modal vector coefficients and associated modal scaling ($A_{pqr} = Q_r \psi_{pr} \psi_{qr}$) did not depend upon the assumption of proportional damping, Equation 7.18 is valid for any damping condition and is the most general form of modal scaling. Note that in general, *modal A* will be complex valued. Equation 7.18 can also be written in an equivalent form that clearly expresses the dependence of *modal A* on the modal vector scaling.

$$M_{A_r} = \frac{1}{Q_r} = \frac{\psi_{pr}\ \psi_{qr}}{A_{pqr}} \tag{7.19}$$

Note that the above development of *modal A* is in terms of displacement over force units. If measurements, and therefore residues are developed in terms of any other units (velocity over force or acceleration over force), Equation 7.19 or the FRF data will have to be altered accordingly.

Consistent with the *modal B* scaling value defined previously from an analytical viewpoint, *modal B* can be determined once *modal A* is known by way of the modal frequency.

$$M_{B_r} \ = \ -\lambda_r\ M_{A_r} \tag{7.20}$$

7.3 Modal Mass, Modal A Units Discussion

In general, the modal vectors are considered to be dimensionless since they represent relative patterns of motion. Therefore, the modal mass or modal A scaling terms are considered to carry the units of the respective measurement. For example, the development of the frequency response is based upon displacement over force units. The residue must therfore, have units of length over force-seconds. Since the modal A scaling coefficient is inversely related to the residue, modal A will have units of force-seconds over length. This unit combination is the same as mass over seconds. Likewise since modal mass is related to modal A, for porportionally damped systems, through a direct relationship involvinng the damped natural frequency, the units on modal mass are mass units as expected.

The following table summarizes the units of modal A and modal mass for typical consistent unit applications.

Consistent Units Relationships				
Mass	Force	Length	Modal A (M_{Ar})	Modal Mass (M_r)
M	F	L	M/S	M
KG	NT	Meter	KG/S	KG
KG	KGf	g-S-S	KG/S	KG
LBm	LBf	g-S-S	LBm/S	LBm
Slug	LBf	Feet	Slug/S	Slug