2. SINGLE DEGREE OF FREEDOM SYSTEM

2.1 Theory

The general mathematical representation of a single degree of freedom system is expressed using Newton's second law in Equation 2.1:

$$M\ \ddot{x}(t)\ +\ C\ \dot{x}(t)\ +\ K\ x(t)\ =\ f(t) \tag{2.1}$$

and is represented schematically in Figure (2-1).

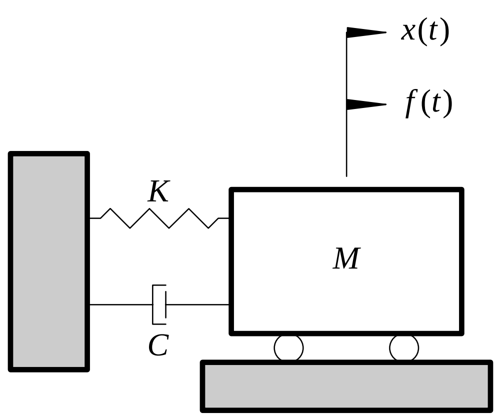

Figure 2-1. Single Degree of Freedom System

Equation 2.1 is a linear, time invariant, second order differential equation. The total solution to this problem involves two parts as follows:

$$x(t) = x_c(t) + x_p(t)$$

where:

- $x_c(t)$ = Transient portion
- $x_p(t)$ = Steady state portion

By setting $f(t) = 0$, the homogeneous (transient) form of Equation 2.1 can be solved.

$$M\, \ddot{x}(t) + C\, \dot{x}(t) + K\, x(t) = 0 \tag{2.2}$$

From differential equation theory, the solution can be assumed to be of the form $x_c(t) = Xe^{st}$, where s is a constant to be determined. Taking appropriate derivatives and substituting into Equation 2.2 yields:

$$(Ms^2 + C\, s + K)\; X(s)\; e^{s\,t} = 0$$

Thus, for a non-trivial solution $\left(X(s)\; e^{s\,t} \neq 0 \right)$:

$$s^2 + (\, C\, /\, M\,)\, s + (\, K\, /\, M\,) = 0 \tag{2.3}$$

Equation 2.3 is the system's characteristic equation, whose roots λ_1 and λ_2 (λ = system pole) are:

$$\lambda_{1,2} = -\frac{C}{2M} \pm \sqrt{\left(\frac{C}{2M} \right)^2 - \left(\frac{K}{M} \right)}$$

Thus the homogeneous solution of Equation 2.1 is:

$$x_c(t) = X_1 e^{\lambda_1 t} + X_2 e^{\lambda_2 t}$$

where X_1 and X_2 are constants determined from the initial conditions imposed on the system at $t = 0$.

The particular solution (steady state) is a function of the form of the forcing function. If the forcing function is a pure sine wave of a single frequency, the response will also be a sign wave of the same frequency. If the forcing function is random in form, the response is also random.

2.2 Laplace Domain Theory

Equation 2.1 is the time domain representation of the system in Figure (2-1). An equivalent equation of motion may be determined for the Laplace or s domain. This representation has the advantage of converting a differential equation to an algebraic equation. This is accomplished by taking the Laplace transform of Equation 2.1, thus:

$$\mathbf{L}\ \{\ M\ \ddot{x} + C\ \dot{x} + K\ x\ \} = M\ (\ s^2\ X(s) - s\ x(0) - \dot{x}(0)\ )$$

$$+ C\ (\ s\ X(s) - x(0)\ ) + K\ X(s)$$

$$\mathbf{L}\ \{\ M\ \ddot{x} + C\ \dot{x} + K\ x\ \} = (\ M\ s^2 + C\ s + K\ )\ X(s) - Ms\ x(0) - M\ \dot{x}(0) - C\ x(0)$$

$$\mathbf{L}\ \{\ f(t)\ \}\ = F(s)$$

Thus Equation 2.1 becomes:

$$[M\ s^2 + C\ s + K]\ X(s) = F(s) + (M\ s + C)\ x(0) + M\dot{x}(0) \tag{2.4}$$

where:

- $x(0)$ is the initial displacement at time $t = 0$.
- $\dot{x}(0)$ is the initial velocity at time $t = 0$.

If the initial conditions are zero, Equation 2.4 becomes:

$$[M\ s^2 + C\ s + K]\ X(s) = F(s) \tag{2.5}$$

Let $B(s) = M\ s^2 + C\ s + K$. $B(s)$ is referred to as the system impedance. Then Equation 2.5 becomes:

$$B(s)\ X(s) = F(s) \tag{2.6}$$

Equation 2.6 is an equivalent representation of Equation 2.1 in the Laplace domain. The Laplace domain (s domain) can be thought of as complex frequency ($s = \sigma + j\,\omega$). Therefore, the quantities in Equation 2.6 can be thought of as follows:

- $F(s)$ - the Laplace domain (complex frequency) representation of the forcing function $f(t)$
- $X(s)$ - the Laplace domain (complex frequency) representation of the system response $x(t)$

Equation 2.6 states that the system response $X(s)$ is directly related to the system forcing function $F(s)$ through the quantity $B(s)$. If the system forcing function $F(s)$ and its response $X(s)$ are known, $B(s)$ can be calculated. That is:

$$B(s) = \frac{F(s)}{X(s)}$$

More frequently one would like to know what the system response is going to be due to a known input $F(s)$, or:

$$X(s) = \frac{F(s)}{B(s)} \tag{2.7}$$

By defining $H(s) = \dfrac{1}{B(s)}$, Equation 2.7 becomes:

$$X(s) = H(s)\,F(s) \tag{2.8}$$

The quantity $H(s)$ is known as the system *transfer function*. In other words, a transfer function relates the Laplace transform of the system input to the Laplace transform of the system response. From Equations 2.5 and 2.8, the transfer function can be defined as:

$$H(s) = \frac{X(s)}{F(s)} = \frac{1/M}{s^2 + (C/M)\,s + (K/M)} \tag{2.9}$$

assuming initial conditions are zero.

The denominator term is referred to as the system characteristic equation. The roots of the characteristic equation are:

$$\lambda_{1,2} = -\,(C / 2M) \pm \sqrt{ (C / 2M)^2 - (K / M) } \tag{2.10}$$

Note that the above definition of the transfer function establishes a form of an analytical model that can be used to describe the transfer function. This analytical model involves a numerator and denominator polynomial with scalar coefficients. For the single degree of freedom case, the numerator polynomial is zeroeth order and the denominator polynomial is second order.

2.3 Definition Of Terms

2.3.1 Critical Damping

Critical damping C_c is defined as being the damping which reduces the radical in the characteristic equation to zero.

$$(C_c / 2M)^2 - (K / M) = 0$$

$$(C_c / 2M) = \sqrt{K / M} = \Omega_1$$

$$C_c = 2M\ \Omega_1 = \text{critical damping coefficient}$$

$$\Omega_1 = \text{undamped natural frequency } (rad / sec)$$

2.3.2 Fraction of Critical Damping - Damping Ratio (Zeta)

The fraction of critical damping or damping ratio, ζ, is the ratio of the actual system damping to the critical system damping.

$$\zeta_1 = C / C_c$$

The roots of characteristic Equation 2.10 can now be written as:

$$\lambda_{1,2} = \left(-\zeta_1 \pm \sqrt{\zeta_1^{\,2} - 1} \right) \Omega_1 \tag{2.11a}$$

$$\lambda_{1,2} = \left(-\zeta_1 \pm j\, \sqrt{1 - \zeta_1^{\,2}} \right) \Omega_1 \tag{2.11b}$$

2.4 System Classification

Systems can be classified depending on their damping ratios. That is:

- Overdamped system: $\zeta_1 > 1$
- Critically damped system: $\zeta_1 = 1$
- Underdamped system: $\zeta_1 < 1$

Figures (2-2) through (2-4) illustrate typical the time domain response of these 3 different cases. The following plots illustrate the location of the roots of the characteristic equation in the *s*-plane. Figure (2-2) (overdamped) shows two real roots that lie on the σ axis, if damping were to increase the roots would move apart. For Figure (2-3) (critically damped), there are two identical real roots. For Figure (2-4) (underdamped), there are two complex roots, complex conjugates of each other. As the damping and/or the frequency changes, these roots stay in the second and third quadrant of the graph. It should be pointed out that if any roots of the characteristic equation lie to the right of the $j\,\omega$ axis, the system would be unstable.

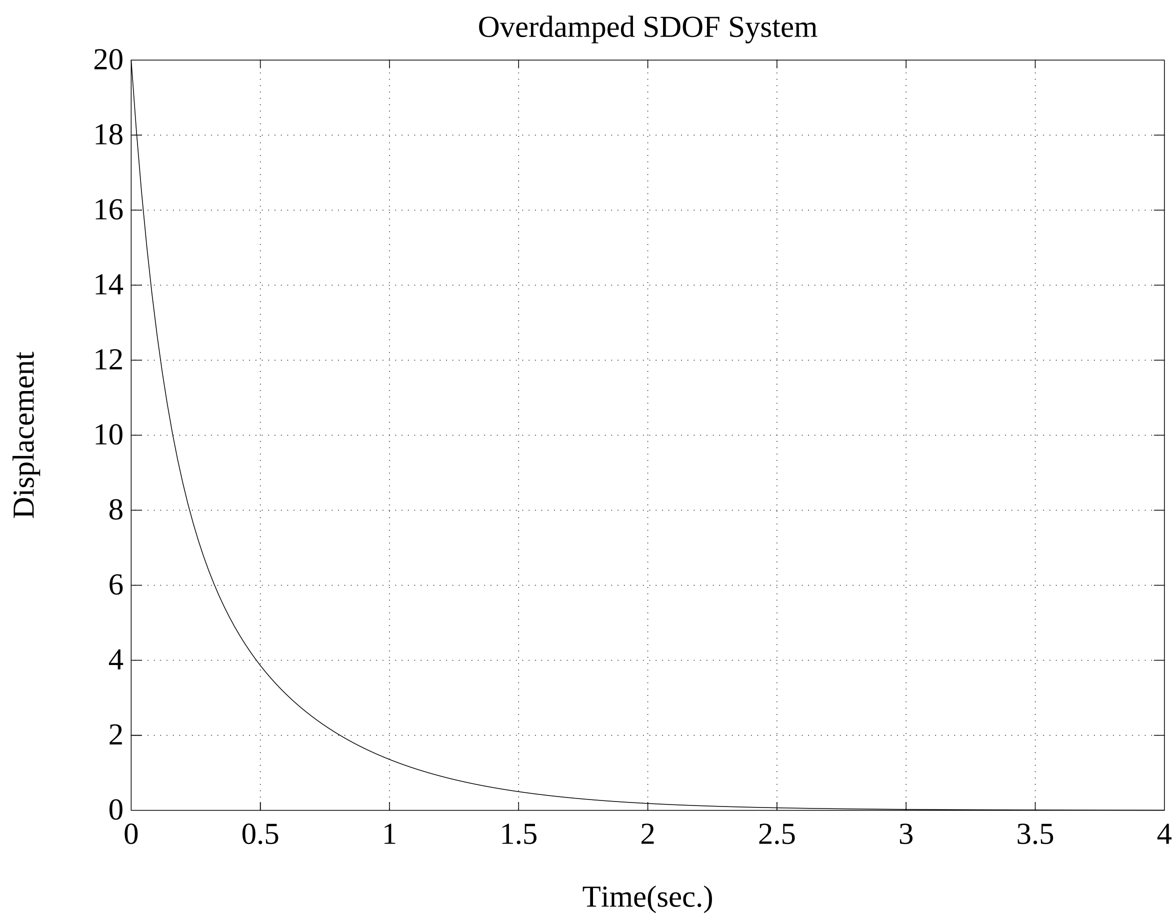

Figure 2-2. Overdamped SDOF System Response: Initial Displacement

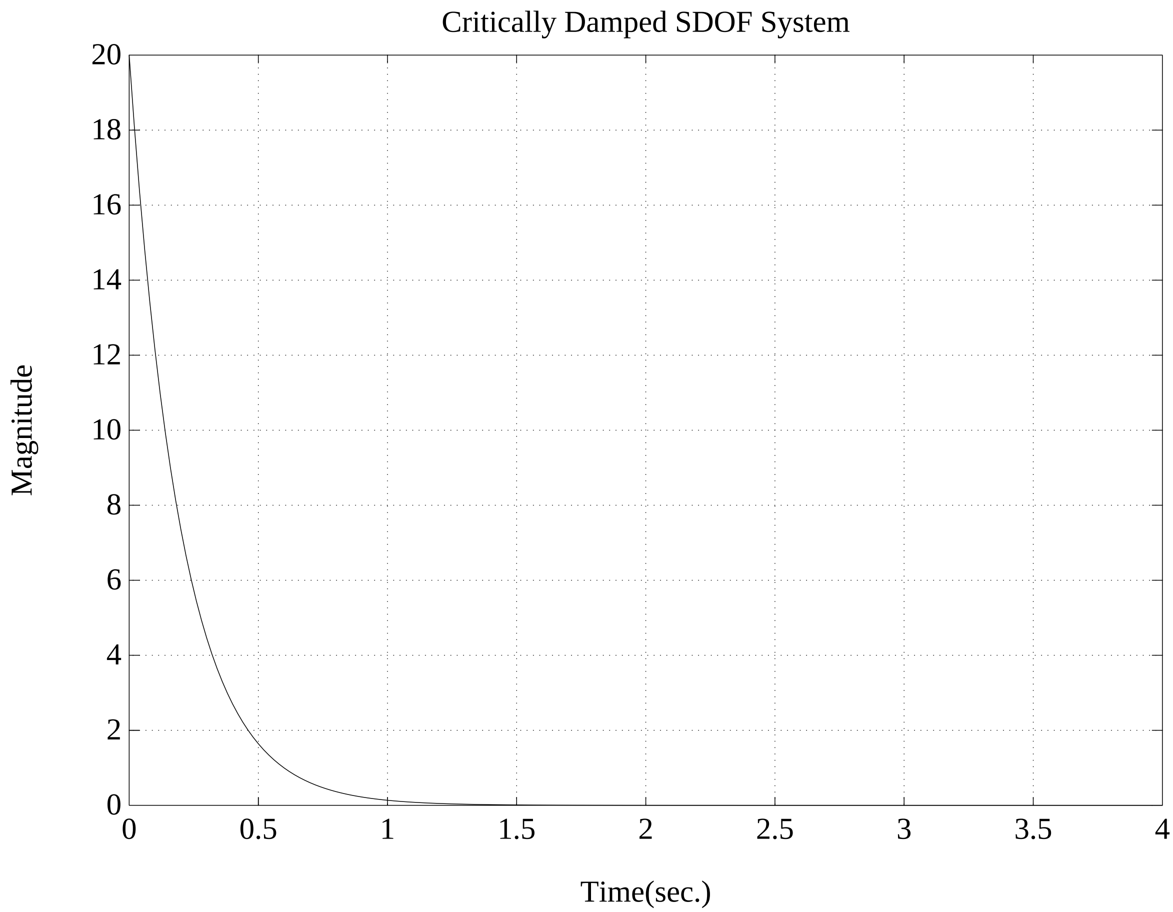

Figure 2-3. Critically Damped SDOF System Response: Initial Displacement

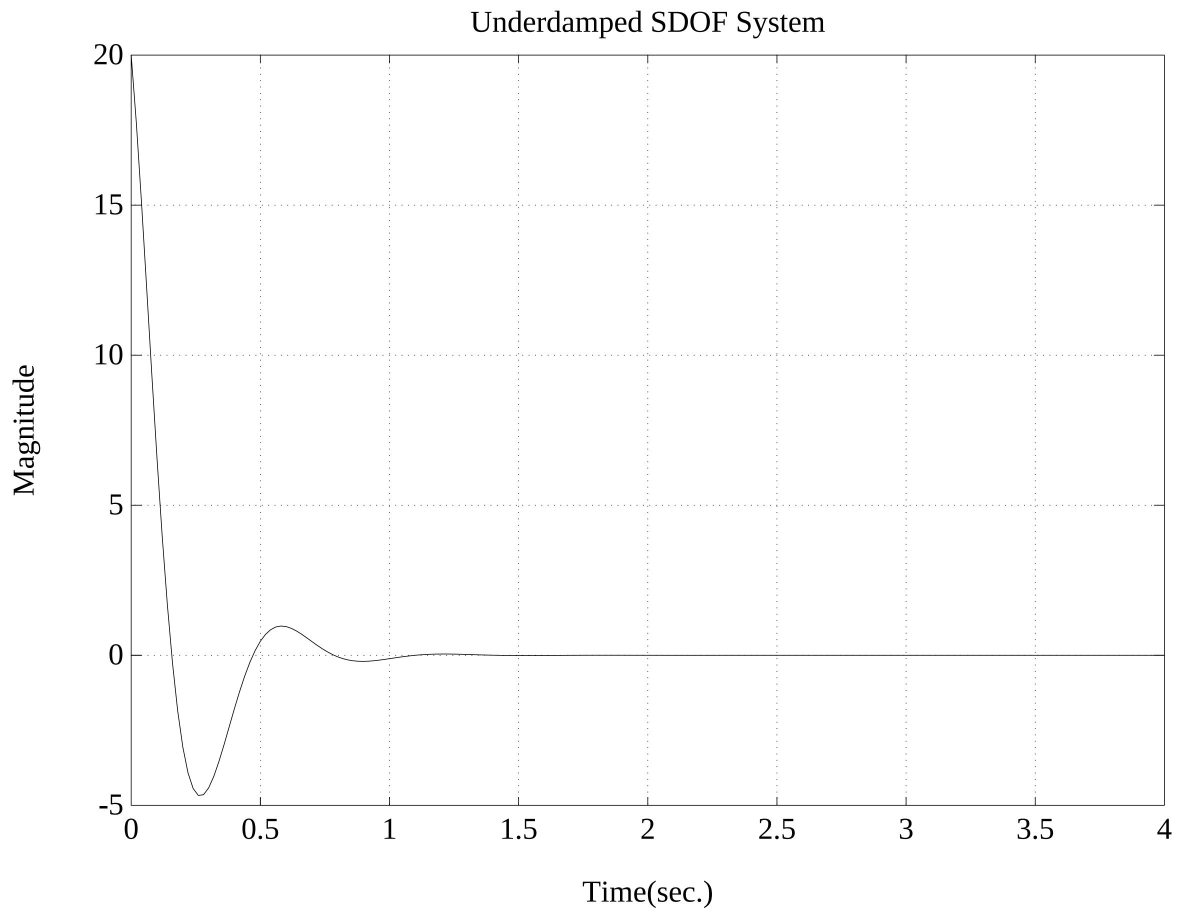

Figure 2-4. Underdamped SDOF System Response: Initial Displacement

Figures (2-5) through (2-7) show the location of the roots, in the s or Laplace plane, of the characteristic equation for each of the three cases.

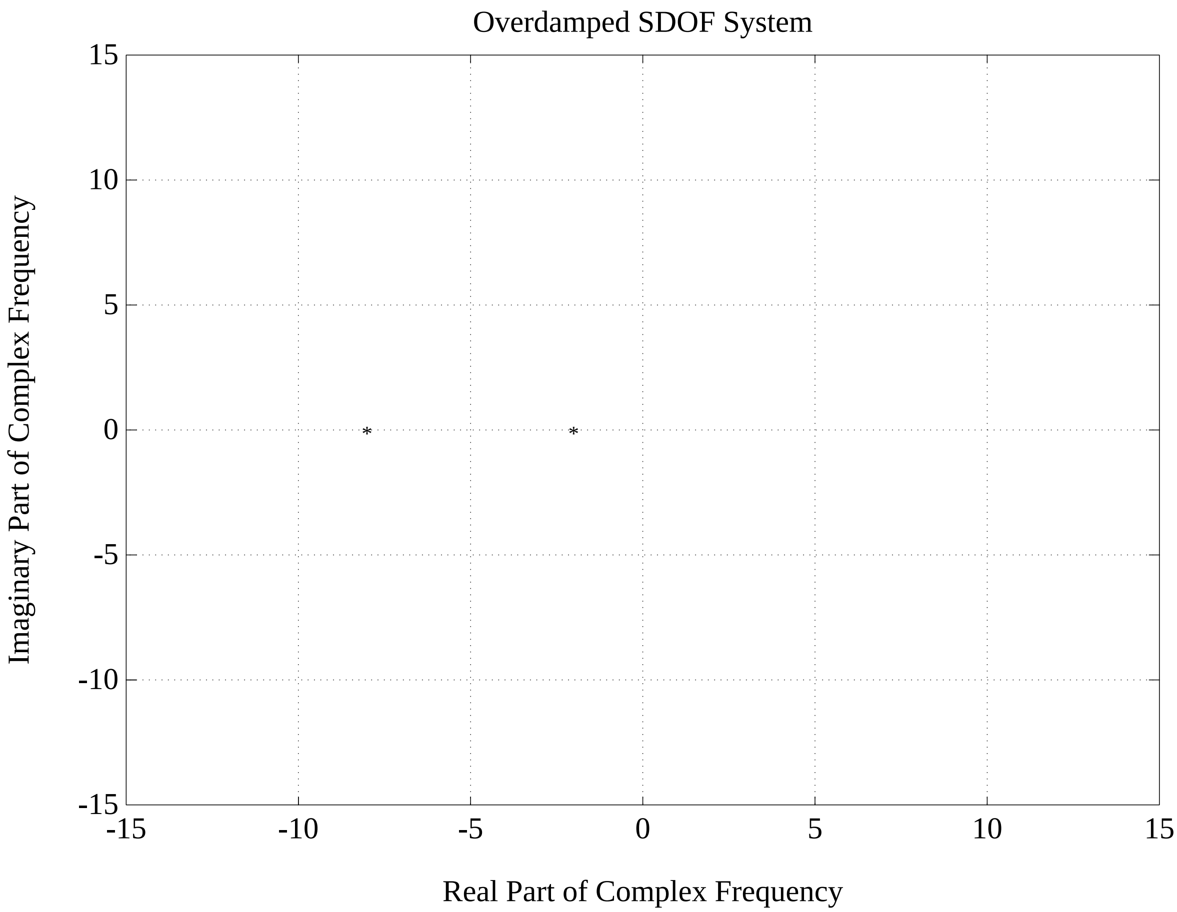

Figure 2-5. Overdamped SDOF System

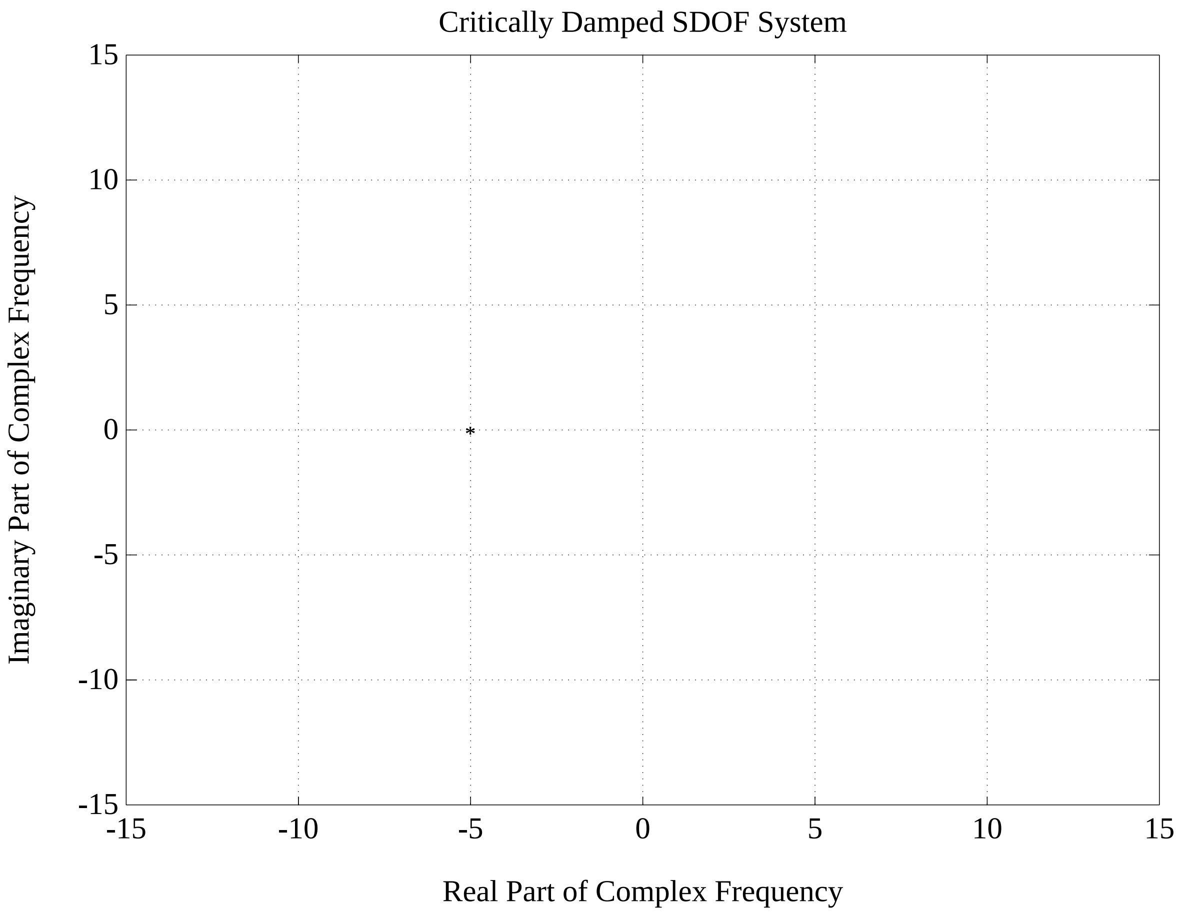

Figure 2-6. Critically Damped SDOF System

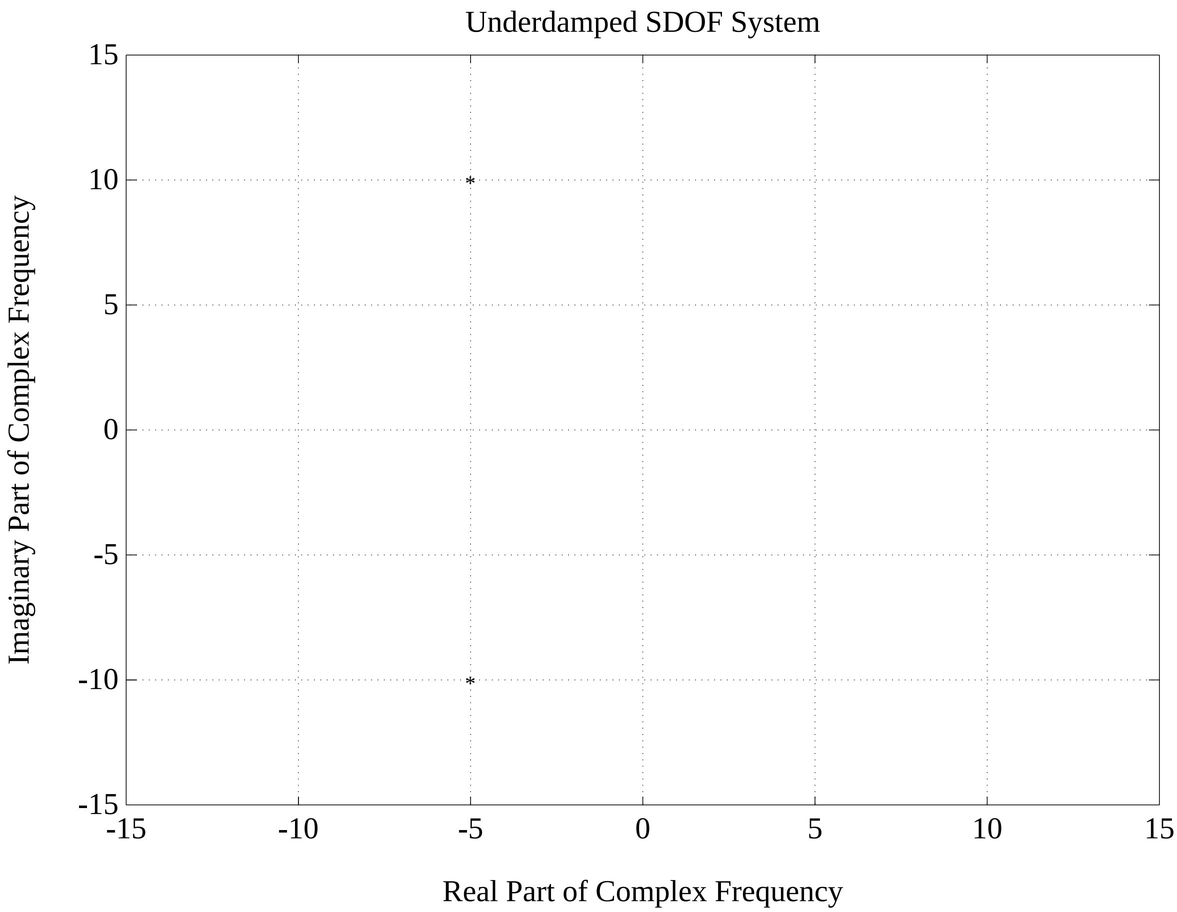

Figure 2-7. Underdamped SDOF System

For most real structures, unless active damping systems are present, the damping ratio is rarely greater than ten percent. For this reason, all further discussion will be restricted to underdamped systems $\zeta_1 < 1$. For an underdamped system, the roots of the characteristic equation can be written as:

$$\lambda_{1,2} = \sigma_1 \pm j\omega_1$$

where:

- $\sigma_1 =$ damping factor (units of rad/sec)

- ω_1 = damped natural frequency

Note that for this case λ_2 is always the complex conjugate of λ_1. Therefore, the λ_2 notation will be replaced in further equations by λ_1^*.

Using Equation 2.11 the above parameters can be related to the damping ratio (ζ_1) and the undamped natural frequency (Ω_1) as follows:

$$\zeta_1 = - \frac{\sigma_1}{\sqrt{\omega_1^2 + \sigma_1^2}}$$

$$\sigma_1 = - \zeta_1 \, \Omega_1$$

$$\Omega_1 = \sqrt{\omega_1^2 + \sigma_1^2}$$

The transfer function $H(s)$ can now be rewritten as a product of the roots (in factored) form as follows:

$$H(s) = \frac{1/M}{(\, s - \lambda_1 \,)\,(s - \lambda_1^* \,)} \tag{2.12}$$

where:

- λ = pole of the transfer function
- $\lambda_1 = \sigma_1 + j\omega_1$
- $\lambda_1^* = \sigma_1 - j\omega_1$

The poles of the single degree of freedom system can also be viewed, looking down on the s-plane as shown in Figure (2-8) :

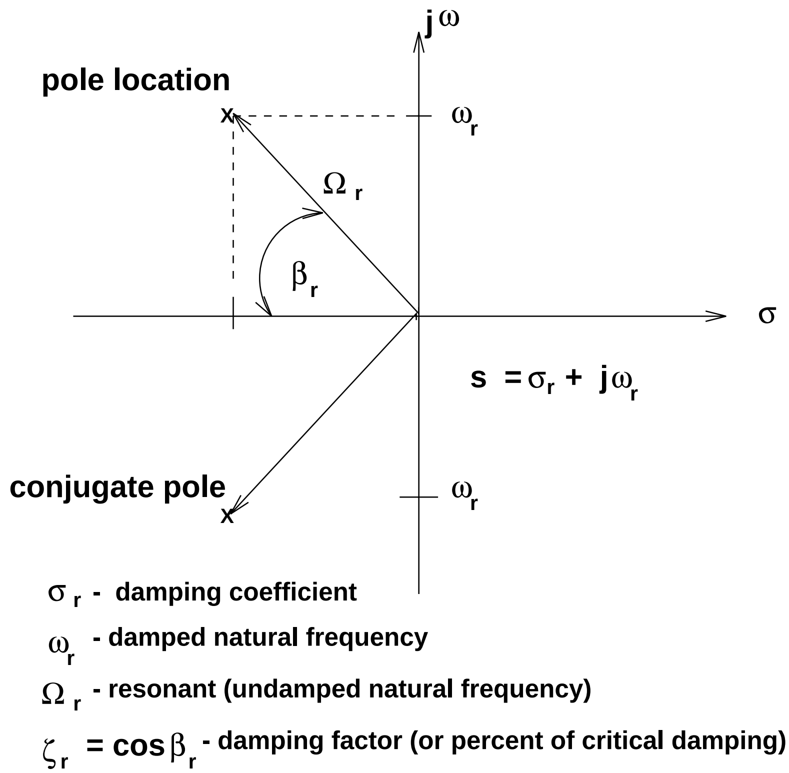

Figure 2-8. Laplace Plane Pole Location

Figures (2-9) through (2-11) illustrate a 3-dimensional plot of Equation 2.12. Figure (2-9) views the surface in real/imaginary format, Figure (2-10) represents the same data in a magnitude/phase format and Figure (2-11) uses a log magnitude/phase format. Remember that the variable s in Equation 2.12 is a complex variable, that is, it has a real part and an imaginary part. Therefore, it can be viewed as a function of two variables which represent a surface. Note also that the FRF measurement that is typically estimated is the slice through these surfaces where $s = j\omega$.

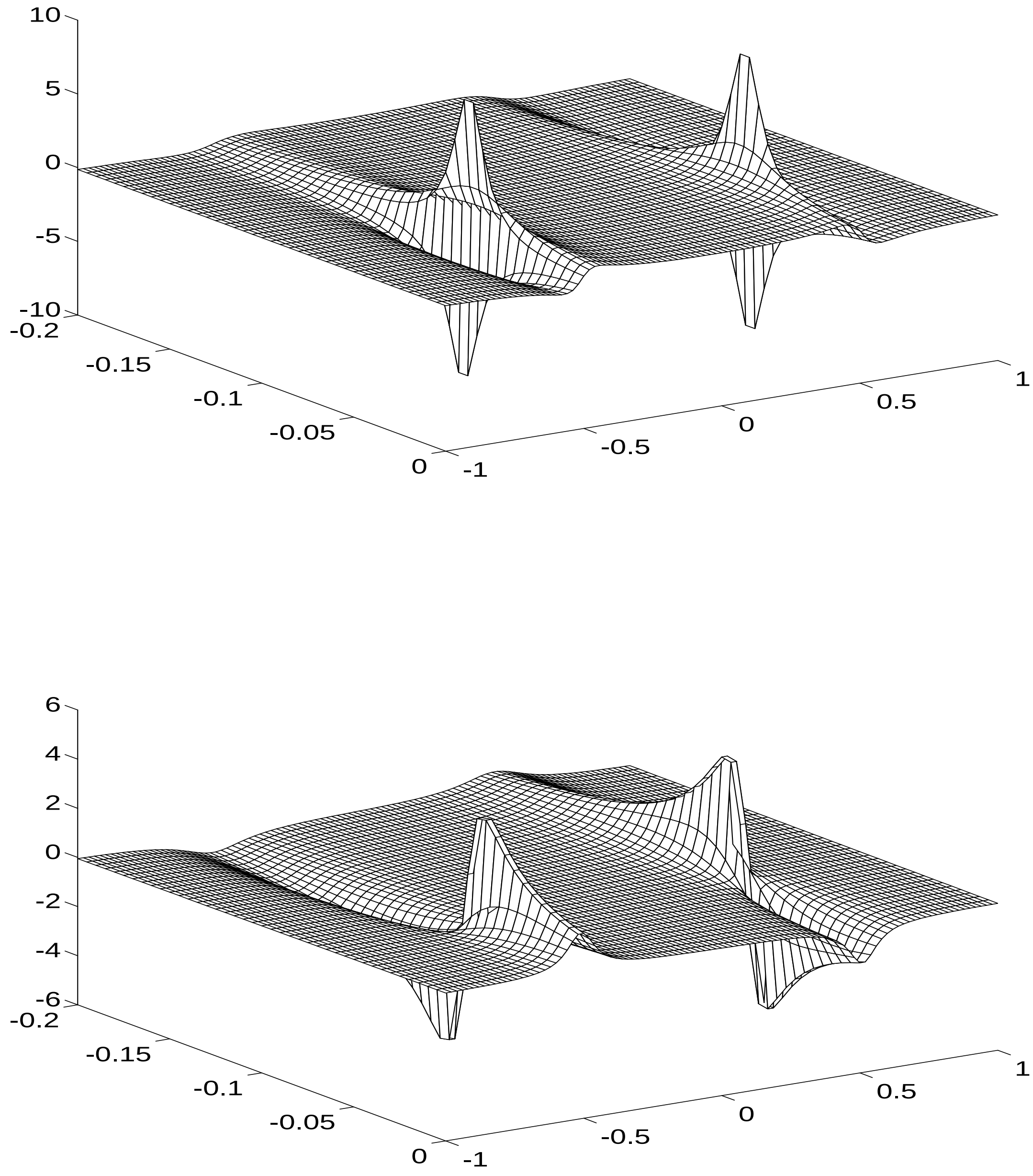

Figure 2-9. Transfer Function (Real-Imaginary), Surface Representation

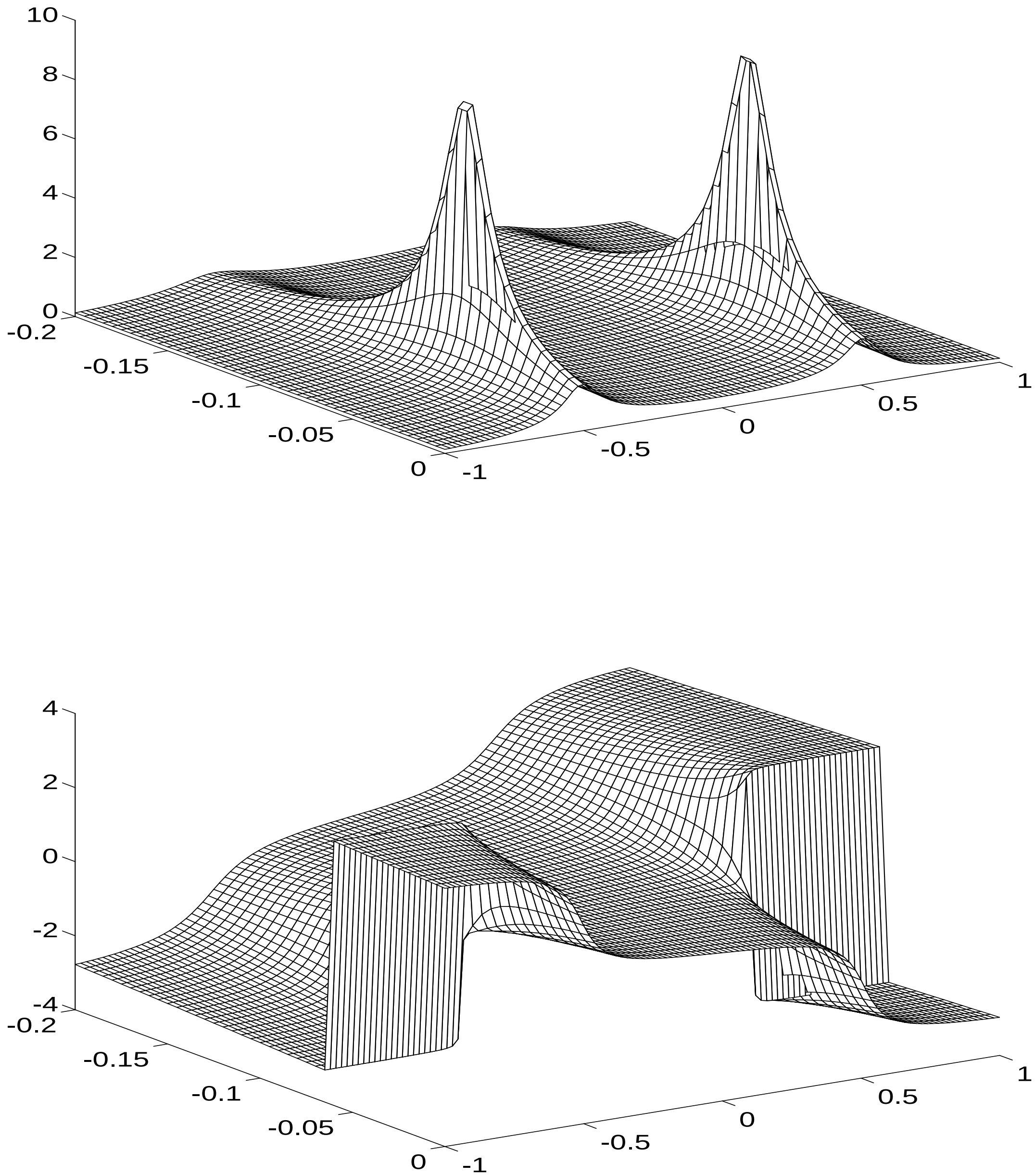

Figure 2-10. Transfer Function (Magnitude-Phase), Surface Representation

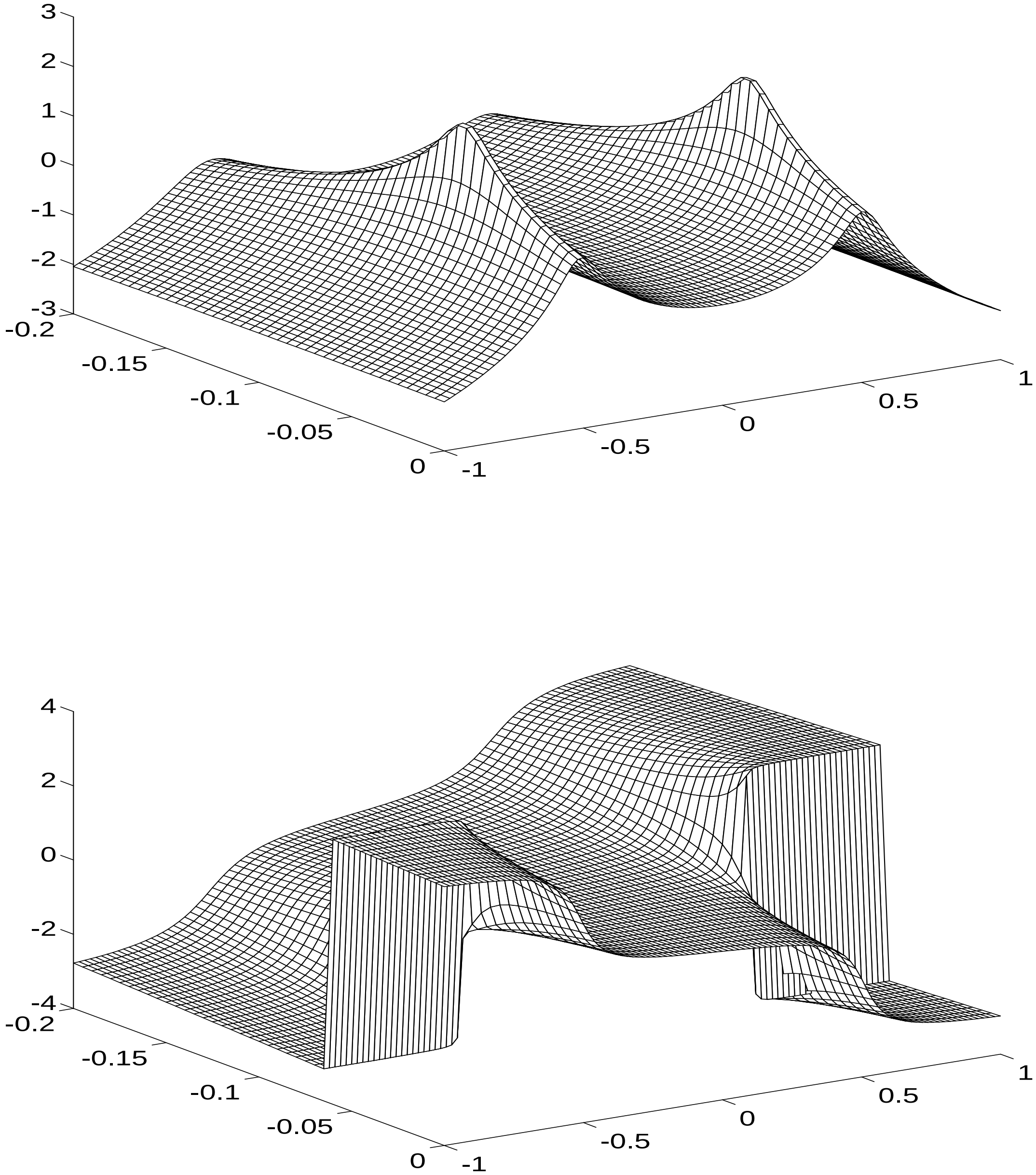

Figure 2-11. Transfer Function (Log Magnitude-Phase), Surface Representation

2.5 Analytical Model - Scalar Polynomial

One popular method of representing the transfer function involves a scalar polynomial representation in the numerator and denominator. For the single degree of freedom case, this is very simple concept that is directly based upon the physical characteristics (M,C,K) of the system. Generalizing Equation (2.9) yields:

$$H(s) = \frac{\beta_0}{\alpha_2\,(s)^2 + \alpha_1\,(s)^1 + \alpha_0\,(s)^0} \tag{2.13}$$

This can be rewritten:

$$H(s) = \frac{\beta_0}{\sum_{k=0}^{2} \alpha_k\,(s)^k} \tag{2.14}$$

This model serves as the basis for many modal parameter estimation methods and is a common formulation utilzed in control theory applications..

2.6 Analytical Model - Partial Fraction

The concept of residues can now be discussed in terms of the partial fraction expansion of the transfer function equation. This is just one common approach to determining the residues. Another popular method involves a polynomial representation in the numerator and denominator.

Equation 2.12 can be expressed in terms of partial fractions:

$$H(s) = \frac{1/M}{(s-\lambda_1)\,(s-\lambda_1^*)} = \frac{c_1}{(s-\lambda_1)} + \frac{c_2}{(s-\lambda_1^*)} \tag{2.15}$$

The residues of the transfer function are defined as being the constants c_1 and c_2 . The terminology and development of residues comes from the evaluation of analytic functions in complex analysis. As will be shown later, the residues of the transfer function are directly related to the amplitude of the impulse response function. The constants c_1 and c_2 (residues) can be found by multiplying both sides of Equation 2.15 by $(s - \lambda_1)$ and evaluating the result at $s = \lambda_1$. Thus:

$$\frac{1/M}{(s - \lambda_1^*)} \Big|_{s = \lambda_1} = c_1 + \left[\frac{c_2 (s - \lambda_1)}{(s - \lambda_1^*)} \right] \Big|_{s = \lambda_1}$$

$$\frac{1/M}{(\lambda_1 - \lambda_1^*)} = c_1$$

Thus:

$$c_1 = \frac{1/M}{(\sigma_1 + j\ \omega_1) - (\sigma_1 - j\omega_1)} = \frac{1/M}{j\ 2\ \omega_1} = A_1$$

Similarly:

$$c_2 = \frac{1/M}{-j2\omega_1} = A_1^*$$

In general, for a multiple degree of freedom system, the residue A_1 can be a complex quantity. But, as shown for a single degree of freedom system A_1 is purely imaginary.

Therefore:

$$H(s) \ = \ \frac{A_1}{(\ s - \lambda_1\ )} \ + \ \frac{A_1^*}{(\ s - \lambda_1^*\ )} \tag{2.16}$$

2.7 Frequency Response Function Representation

The frequency response function is the transfer function (surface) evaluated along the $j\ \omega$ (frequency) axis. Thus, from the previously dervied equations:

Polynomial Model

$$H(s)|_{s=j\omega} = H(\omega) = \frac{\beta_0}{\alpha_2\,(j\omega)^2 + \alpha_1\,(j\omega)^1 + \alpha_0\,(j\omega)^0} \tag{2.17a}$$

Partial Fraction Model

$$H(s)|_{s=j\omega} = H(\omega) = \frac{A_1}{(j\omega - \lambda_1)} + \frac{A_1^*}{(j\omega - \lambda_1^*)}$$

$$H(\omega) = \frac{A_1}{(j\omega - \sigma_1 - j\omega_1)} + \frac{A_1^*}{(j\omega - \sigma_1 + j\omega_1)}$$

$$H(\omega) = \frac{A_1}{j(\omega - \omega_1) - \sigma_1} + \frac{A_1^*}{j(\omega + \omega_1) - \sigma_1} \tag{2.17b}$$

From an experimental point of view, when one talks about measuring a transfer function, the frequency response function is actually being measured.

The value of the frequency response function at the damped natural frequency of the system is:

$$H(\omega_1) = -\ \frac{A_1}{\sigma_1} + \frac{A_1^*}{j\ 2\ \omega_1 - \sigma_1} \tag{2.18}$$

which can be approximated as:

$$H(\omega_1) = -\ \frac{A_1}{\sigma_1}$$

The second term on the right of Equation 2.18 approaches zero as ω_1 gets large. In other words, the contribution of the negative frequency portion of the frequency response function is negligible.

Therefore, many single degree of freedom models are represented as:

$$H(\omega) \approx \frac{A_1}{(\ j\omega - \lambda_1)} \tag{2.19}$$

Another way of interpreting Equation 2.17 is that the value of the transfer function, for a single degree of freedom system, at a particular frequency (ω) is a function of the residue, damping, and damped natural frequency.

2.8 Impulse Response Function Representation

The impulse response function of the single degree of freedom system can be determined from Equation 2.17 assuming that the initial conditions are zero and that $F(s) = 1$ for a system impulse. Thus:

$$X(s) = \frac{A_1}{(s - \lambda_1)} + \frac{A_1^*}{(s - \lambda_1^*)}$$

$$x(t) = \mathbf{L}^{-1} \{ X(s) \}$$

$$x(t) = A_1 e^{\lambda_1 t} + A_1^* e^{\lambda_1^* t} = h(t) = \text{impulse response}$$

$$x(t) = e^{\sigma_1 t} \left[A_1 e^{j\,\omega_1 t} + A_1^* e^{-j\,\omega_1 t} \right]$$

Thus, using Euler's formula for $e^{j\omega_1 t}$ and $e^{-j\omega_1 t}$, the residue A_1 controls the initial amplitude of the impulse response, the real part of the pole is the decay rate and the imaginary part of the pole is the frequency of oscillation. Figures (2-12) and (2-13) illustrate the frequency response and impulse response functions respectively, for a single degree of freedom system.

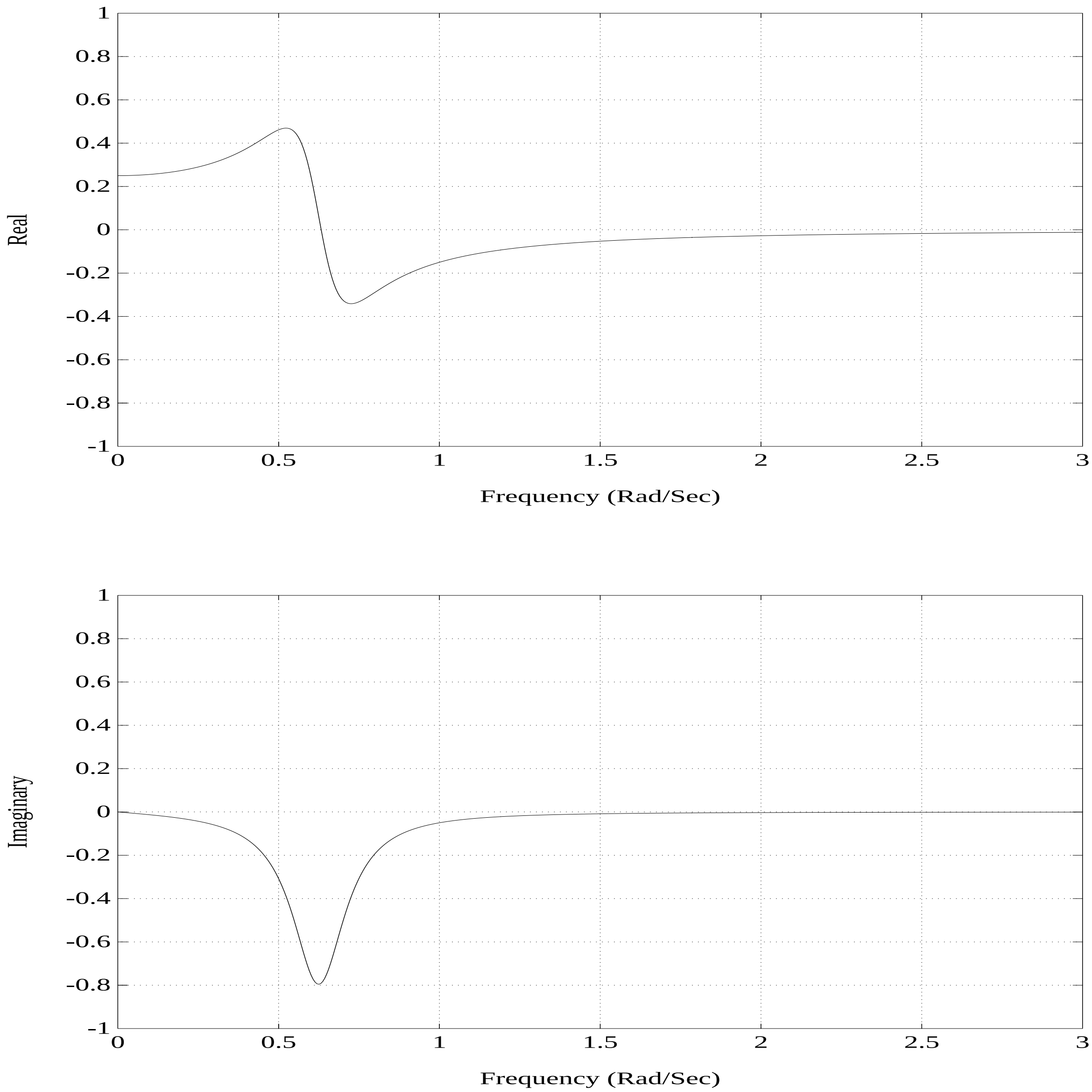

Figure 2-12. Frequency Response Function: SDOF System

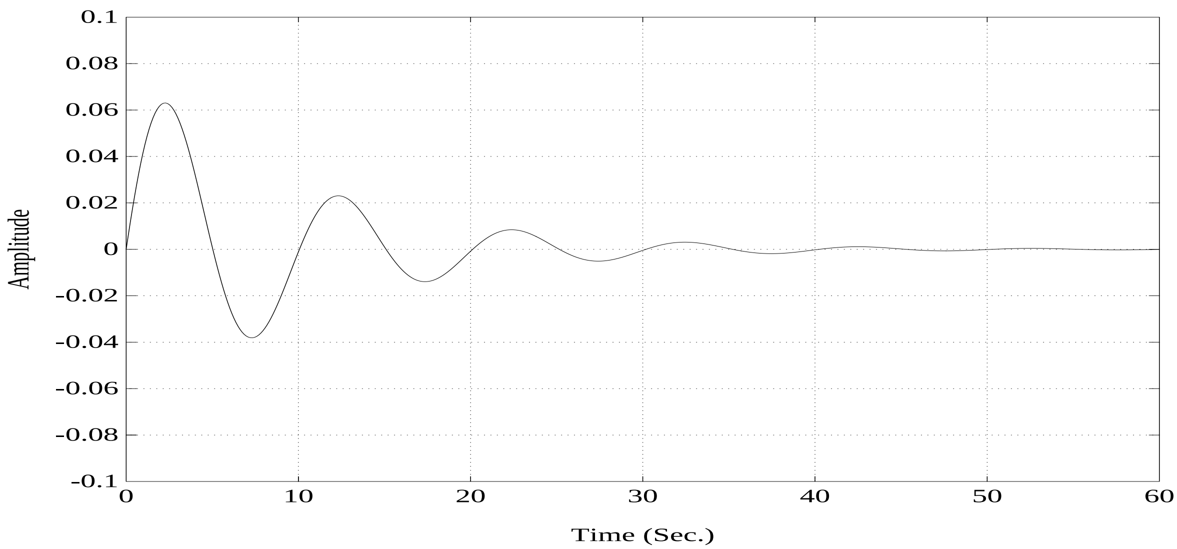

Figure 2-13. Impulse Response Function: SDOF System

2.9 Change of Physical Parameters

While it is not always possible to alter the physical parameter (mass, stiffness and/or damping) of a system and is very difficult to practically alter one physical parameter (mass, for example) without altering another physical parameter (stiffness, for example), it is still important to understand how a change in physical parameter will affect the system characteristics. Figures (2-14), (2-15) and (2-16) show how the frequency response function will be affected due to a change in one physical parameter at a time.

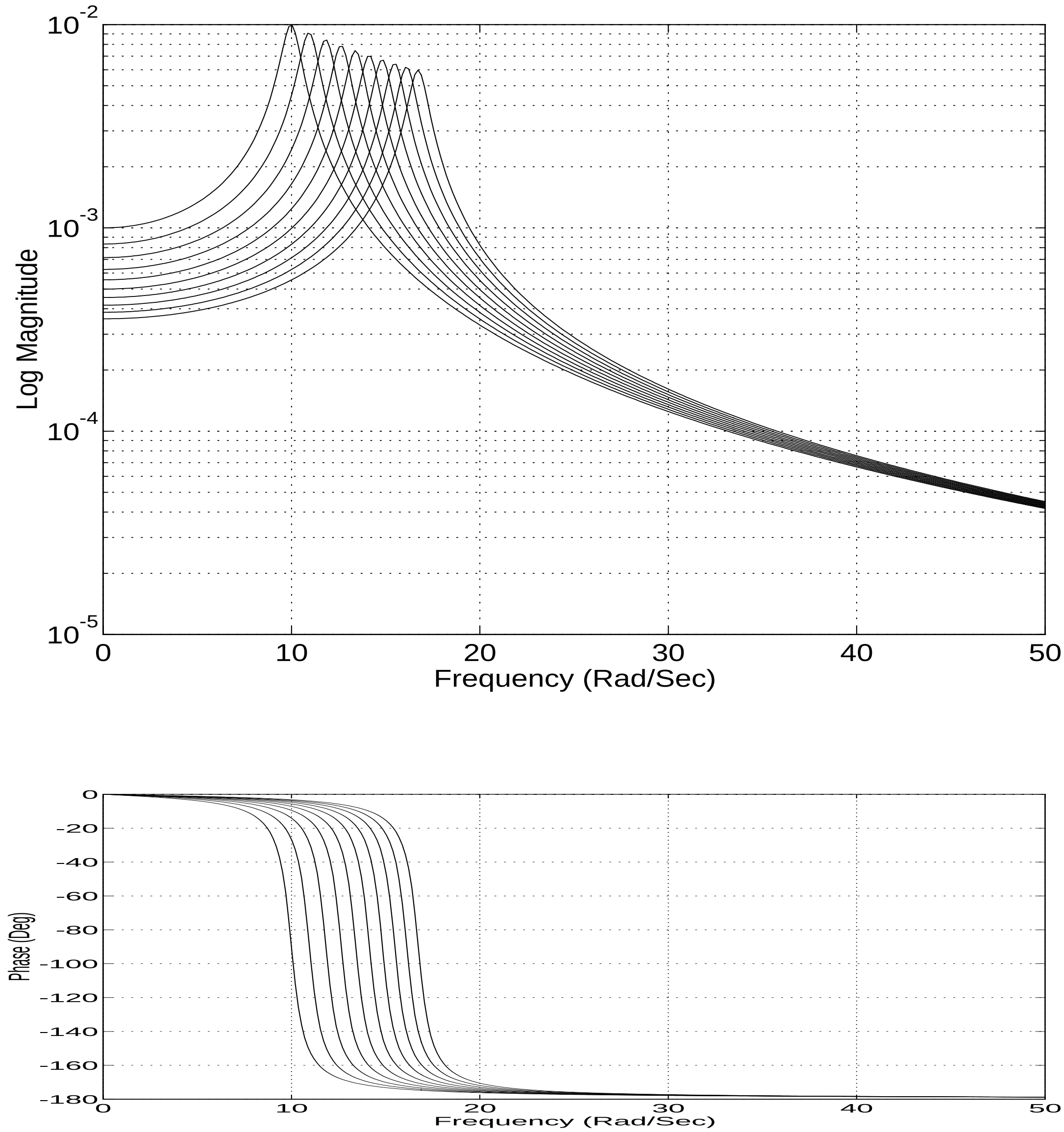

Figure 2-14. Change of Stiffness: SDOF System

Note that a change in stiffness affects both the resonant frequency as well as the system characteristic at low frequency. This dominance of stiffness at low frequency is the reason that this region of the frequency response function is known as the stiffness, or more accurately, compliance line.

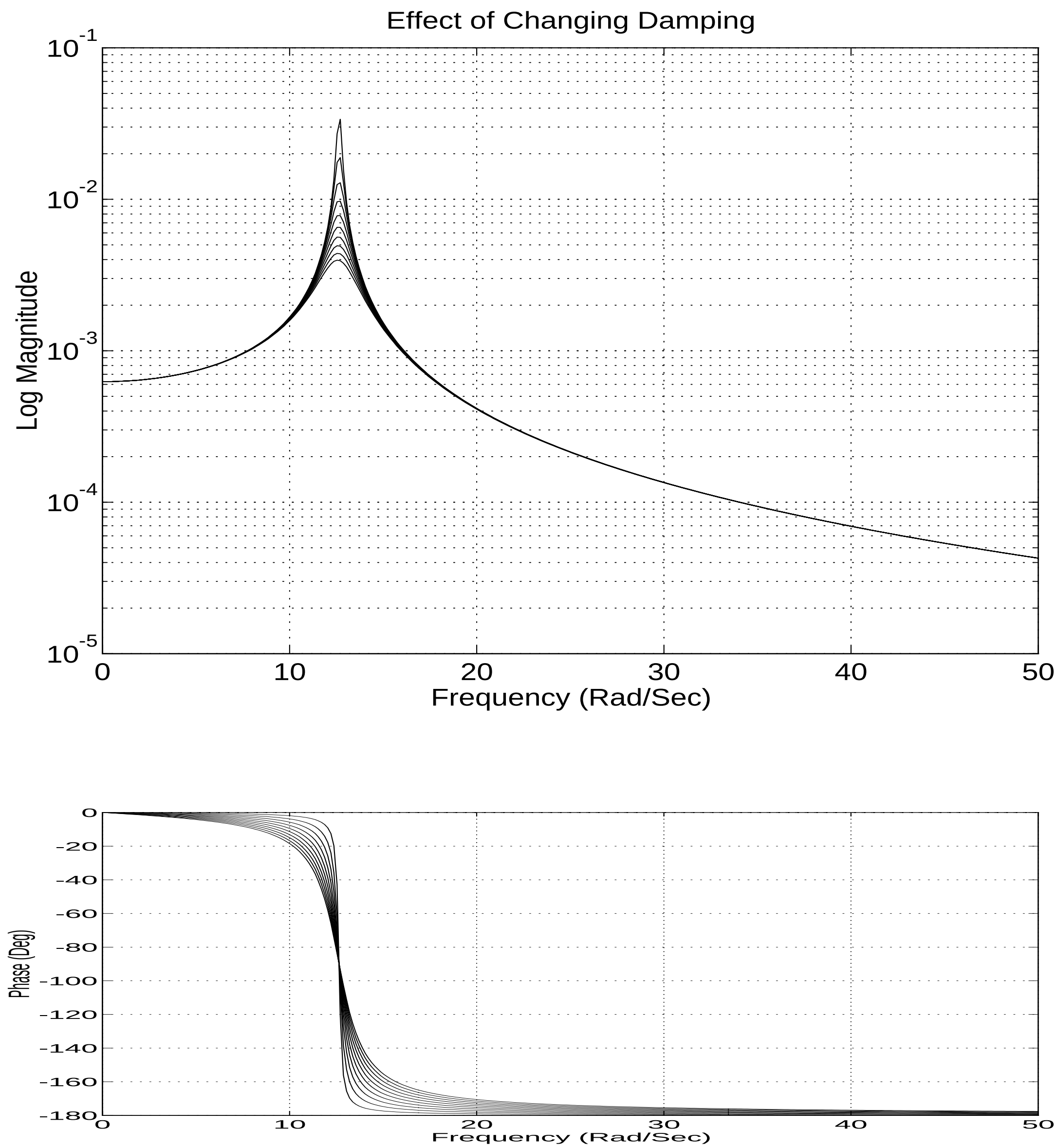

Figure 2-15. Change of Damping: SDOF System

Note that a change in damping has no apparent effect on the resonant frequency. The only noticable change involves a change in frequency response function in the region of the resonant frequency.

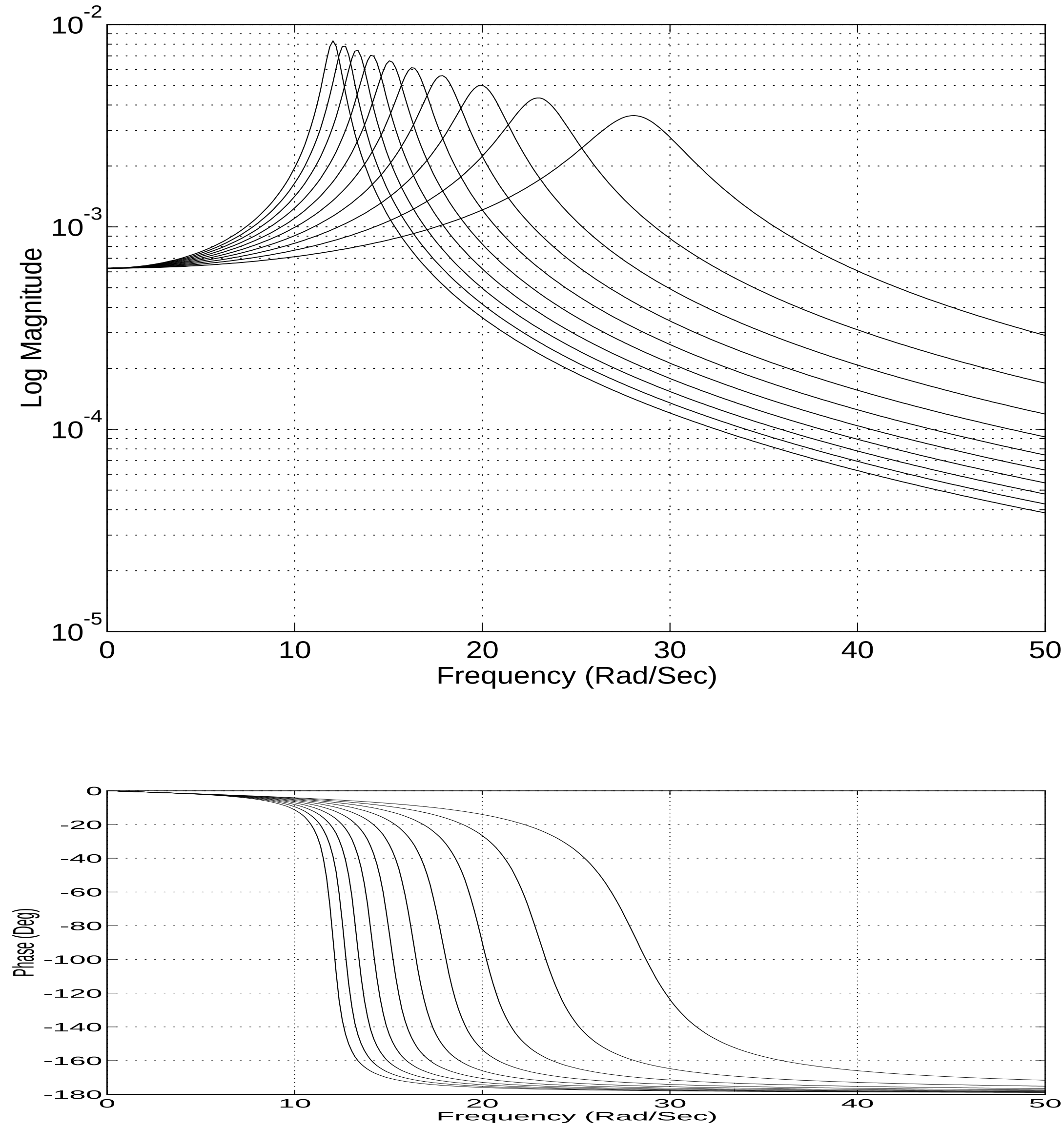

Figure 2-16. Change of Mass: SDOF System

Note that a change in mass affects both the resonant frequency as well as the system characteristic at high frequency. This dominance of mass at high frequency is the reason that this region of the frequency response function is known as the mass line. Also note that as the mass changes, the apparant damping (sharpness of the resonant frequency) changes accordingly. A change in mass affects both the resonant frequency, the system characteristic at high frequency as well as the fraction of critical damping ($\zeta_1 = \dfrac{C}{C_c} = \dfrac{C}{2\ M\ \Omega_1}$).

2.10 Estimating Partial Fraction Parameters

Assuming that a lightly damped single degree of freedom (SDOF) system is being evaluated, the parameters needed for a partial fraction model can be quickly estimated directly from the measured frequency response function. While this approach is based upon a SDOF system, as long as the modal frequencies are not too close together, the method can be used for multiple degree of freedom (MDOF) systems as well.

Starting with the partial fraction model formulation of a SDOF frequency response function

$$H(\omega) = \frac{A_1}{(\ j\omega - \lambda_1\ )} + \frac{A_1^*}{(\ j\omega - \lambda_1^*\ )} \tag{2.20}$$

only the constants λ_1 and A_1 must be estimated. Since $\lambda_1 = \sigma_1 + j\omega_1$, the estimation process begins by estimating ω_1. The damped natural frequency ω_1 is estimated in one of three ways:

- The frequency where the magnitude of the FRF reaches a maximum.
- The frequency where the real part of the FRF crosses zero.
- The frequency where the imaginary part of the FRF reaches a relative minima (or maxima).

Of these three methods, the last approach gives the most reliable results under all conditions.

Once the damped natural frequency ω_1 has been estimated, the real part of the modal frequency, the damping factor σ_1, can be estimated. The damping factor σ_1 can be estimated by using the *half-power bandwidth* method. This method uses the data from the FRF in the region of the resonance frequency to estimate the fraction of critical damping from the following formula:

$$\zeta_1 = \frac{{\omega_b}^2 - {\omega_a}^2}{(2\ \omega_1)^2} \tag{2.21}$$

In the above equation, ω_1 is the damped natural frequency as previously estimated. ω_a is the frequency, below ω_1, where the magnitude is 0.707 of the peak magnitude of the FRF. This corresponds to a half power point. ω_b is the frequency, above ω_1, where the magnitude is 0.707 of the peak magnitude of the FRF. This also corresponds to a half power point.

For lightly damped systems, the above equation can be approximated by the following:

$$\zeta_1 \approx \frac{\omega_b - \omega_a}{2\ \omega_1} \tag{2.22}$$

Once ζ_1 is estimated, the damping factor σ_1 can be estimated from the following equation.

$$\sigma_1 = -\zeta_1\ \Omega_1 \tag{2.23}$$

Again, assuming that the system is lightly damped, $\Omega_1 \approx \omega_1$, the damping factor can be estimated from the following equation:

$$\sigma_1 \approx -\zeta_1\ \omega_1 \tag{2.24}$$

Once the modal frequency λ_1 has been estimated, the residue A_1 can be estimated by evaluating the partial fraction model at a specific frequency. If the specific frequency is chosen to be ω_1, the following result is obtained.

$$H(\omega_1) = \frac{A_1}{(\ j\omega_1 - (\sigma_1 + j\omega_1\ )} + \frac{A_1^*}{(\ j\omega_1 - (\sigma_1 - j\omega_1\ )} \tag{2.25}$$

$$H(\omega_1) = \frac{A_1}{(\ -\sigma_1\ )} + \frac{A_1^*}{(\ 2\ j\ \omega_1 - \sigma_1\ )} \tag{2.26}$$

As long as ω_1 is not too small, the above equation can be approximated by:

$$H(\omega_1) \approx \frac{A_1}{(\ -\sigma_1\ )} \tag{2.27}$$

Therefore, the residue A_1 can be estimated from the following relationship:

$$A_1 \approx (- \sigma_1) H(\omega_1) \tag{2.28}$$

In the above relationship, $H(\omega_1)$ is very close to being a purely imaginary value for the displacement over force FRF. This means that the residue A_1 will be very close to a purely imaginary value as well.

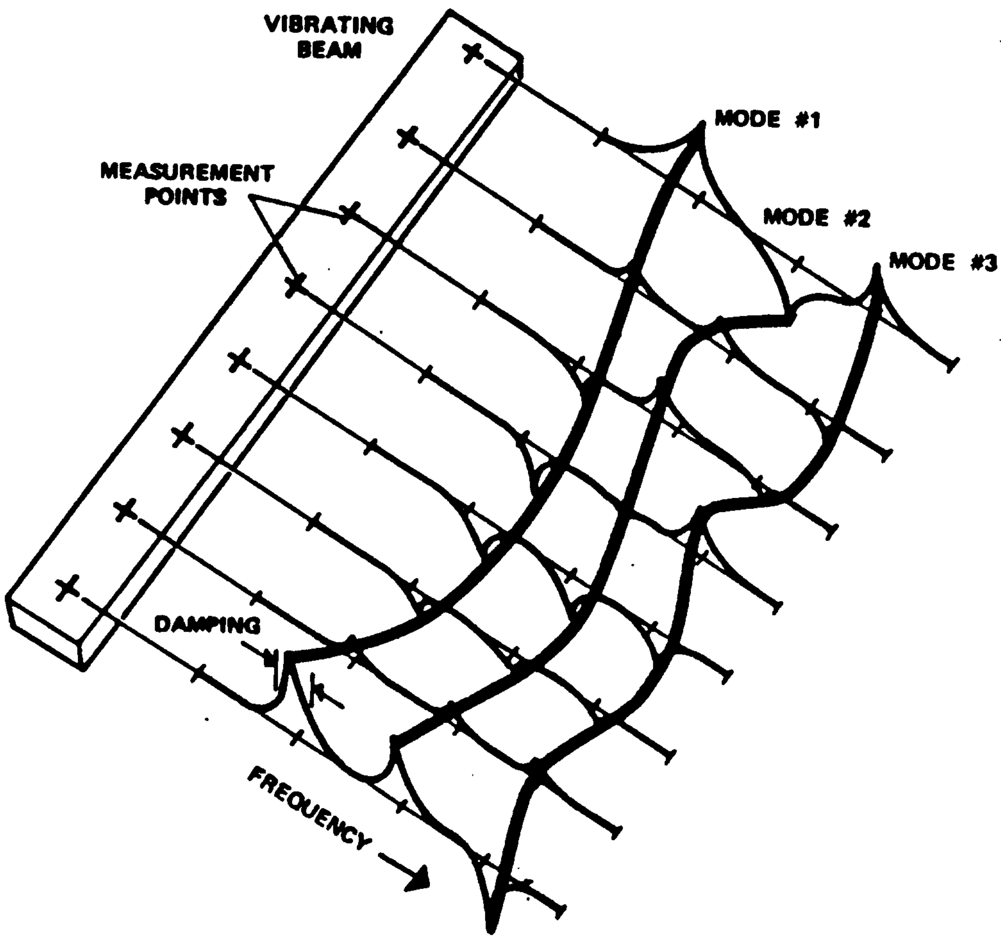

Figure 2-17. Modal Vectors from the Imaginary Part of the FRF

2.10.1 Example

For the following SDOF case, (M=10, K=1600, C=2), the data can be estimated from the FRF as shown in the following plots. The exact answers are λ_1 = -0.1000 + j 12.6487 and A_1 = -j 0.0040. Note that the digitized data, in the neighborhood of the damped natural frequency, is tabulated in Table 2-1.

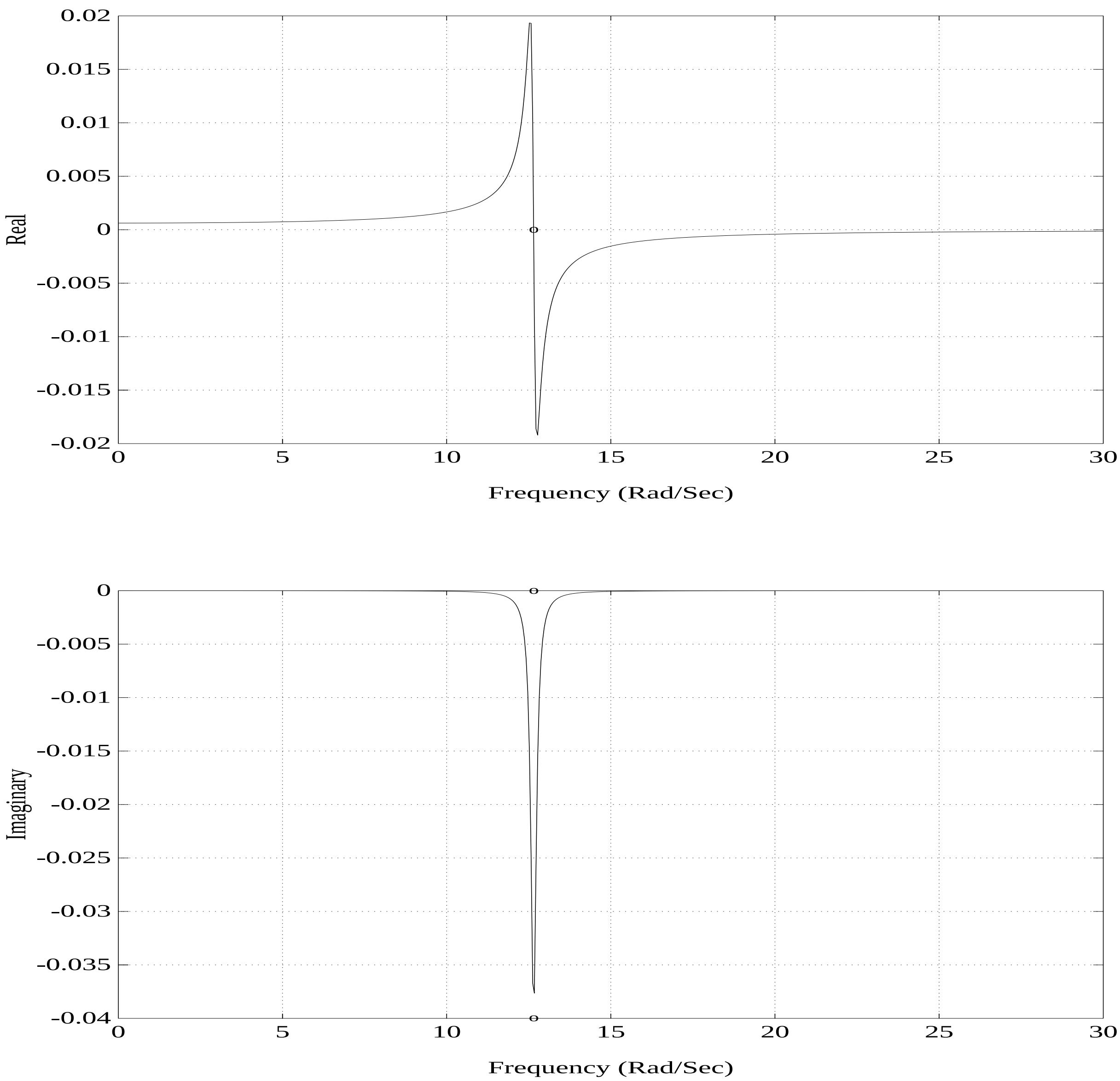

Figure 2-18. Frequency Response Function: SDOF System

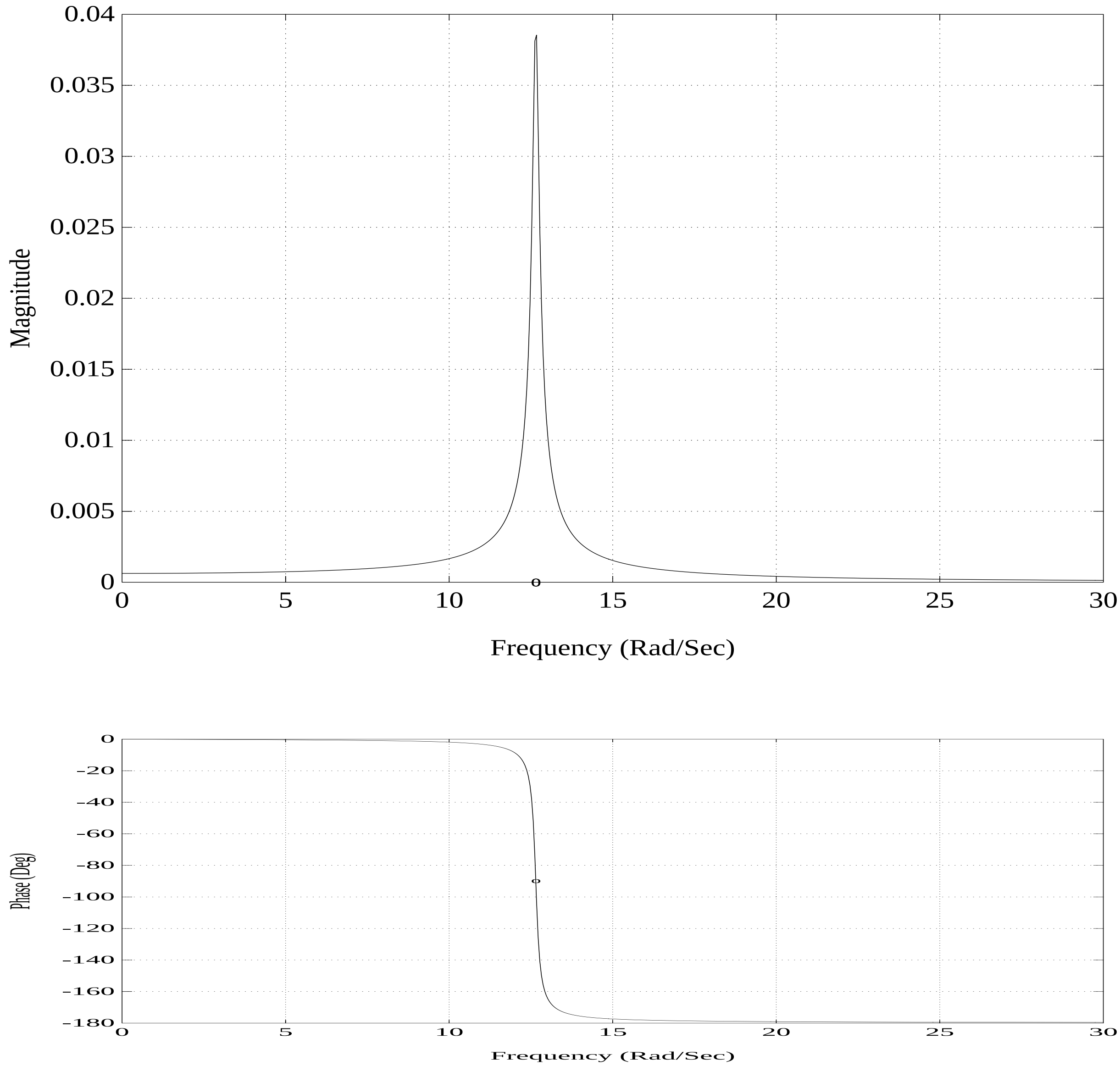

Figure 2-19. Frequency Response Function: SDOF System

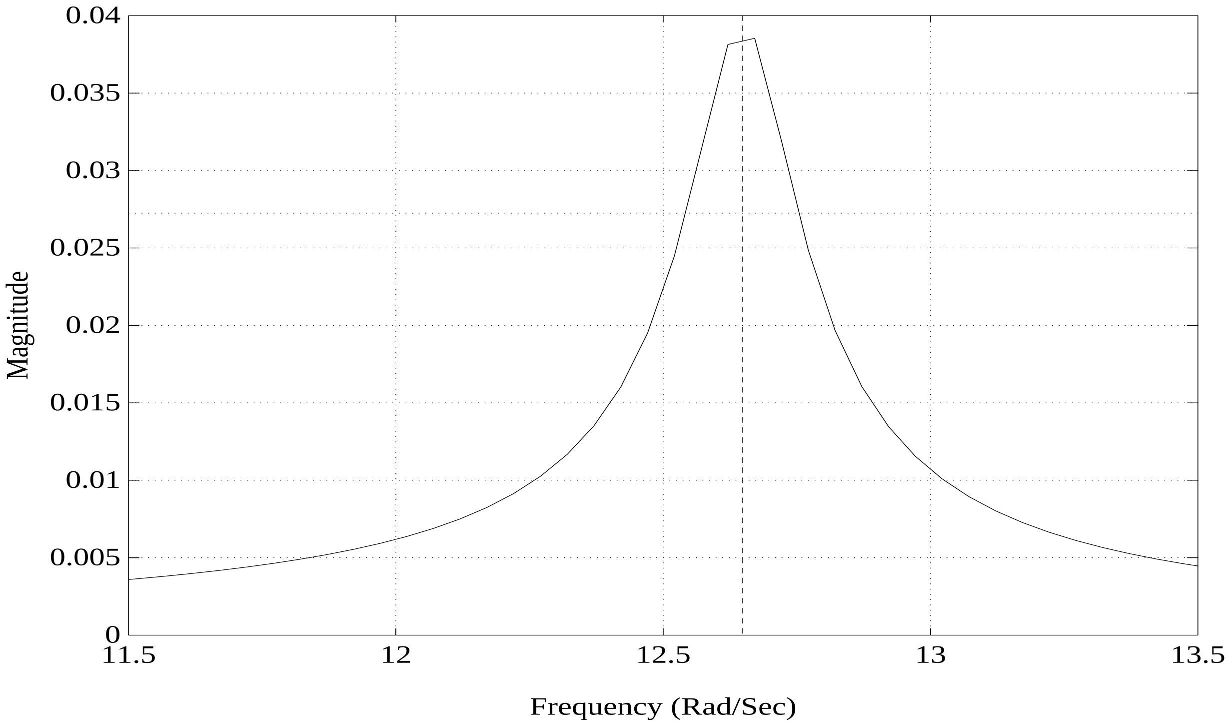

Figure 2-20. Frequency Response Function: SDOF System

Frequency (rad/sec)	Real	Imag	Magnitude	Phase (deg)
12.2705	0.0099	-0.0026	0.0103	-14.578
12.3205	0.0112	-0.0034	0.0117	-16.717
12.3706	0.0127	-0.0045	0.0135	-19.549
12.4207	0.0147	-0.0064	0.0160	-23.452
12.4708	0.0170	-0.0095	0.0195	-29.109
12.5209	0.0193	-0.0150	0.0245	-37.804
12.5710	0.0193	-0.0246	0.0313	-51.903
12.6210	0.0103	-0.0367	0.0381	-74.301
12.6711	-0.0083	-0.0376	0.0385	-102.401
12.7212	-0.0186	-0.0259	0.0319	-125.711
12.7713	-0.0192	-0.0158	0.0249	-140.565
12.8214	-0.0170	-0.0099	0.0197	-149.696
12.8715	-0.0146	-0.0066	0.0160	-155.597
12.9215	-0.0126	-0.0047	0.0135	-159.646
12.9716	-0.0110	-0.0035	0.0115	-162.569

TABLE 2-1. Discrete SDOF Data from Plot

This example also illustrates a common problem with simplified modal parameter estimation. In this example, referring to Figure 2-20, it is apparent that the damped natural frequency occurs between two of the measured frequencies in the frequency respone function. In Figure 2-20, the apparent truncation, or clip, of the frequency response function near the peak frequency is a result of this lack of resolution. For a lightly damped situation, the true magnitude at the resonance may be 2 to 20 times higher. This means that finding the half-power frequencies in order to estimate damping will be impossible since the true magnitude of the resonance is unknown. In this situation, the damping estimate will be in error (too high) which will cause the residue to be in error (too high).

Note that there are a number of other more robust SDOF, and MDOF, modal parameter estimation algorithms that do not require knowledge of the half-power frequencies. These techniques depend only upon the data being accurate at the measured frequencies in order for an accurate estimate of the damping estimate. These methods do not have the accuracy problem of the simplified SDOF case utilized in this example.

2.11 Why Study/Emphasize SDOF Systems?

Frequently, there is some concern that the amount of time studying SDOF systems is not warranted. Many multiple degree-of-freedom (MDOF) systems can be simplified as single degree-of-freedom systems. More often, the MDOF system can be broken down, on a frequency range basis, into frequency regions that are dominated by only one degree-of-freedom. This is the situation with SDOF modal parameter esimation algorithms.

Even though a continuous beam has an infinite number of modes, the evaluation of these modes (estimation of frequency, damping, modal vector and modal scaling) can often be accomplished with essentially single degree-of-freedom (SDOF) concepts. The primary assumption is that each mode of vibration is well separated in frequency from the other modes. This is often the case for lightly damped structures. Different modes of vibration of the beam can be visulaized in Figure 2-17 by noting the solid black line connecting the peaks of the imaginary parts of each frequency response. Normally, these modes are plotted in a wireframe model showing the extrema of the modal vector so that the motion can be easily understood. Figures 2-21 and 2-22 show the first two bending modes of a uniform beam that is pinned at each end.

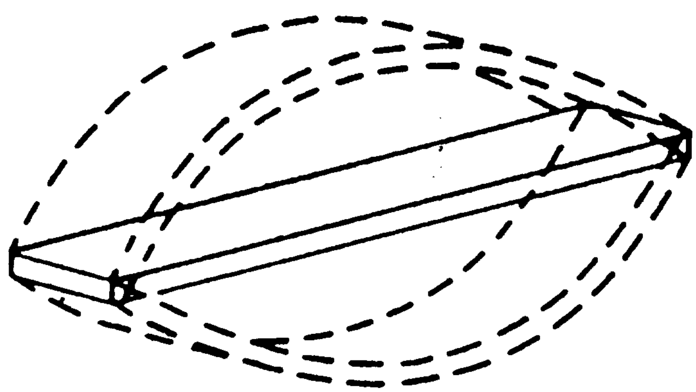

Figure 2-21. First Bending Mode at First Damped Natural Frequency (ω_1)

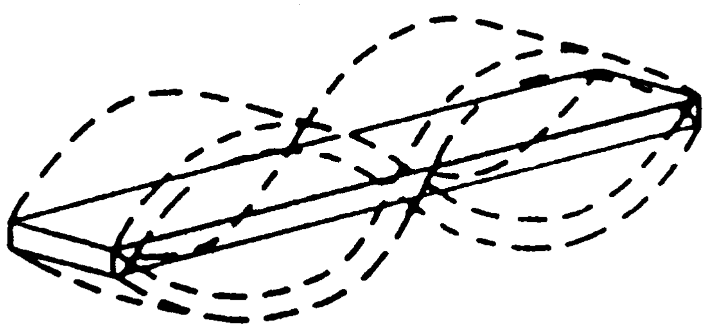

Figure 2-22. Second Bending Mode at Second Damped Natural Frequency (ω_2)

In order to understand why this information can be determined from the imaginary part of the frequency response functions, SDOF theory must be reviewed and extended slightly, primarily from a notational point of view.

The general mathematical representation of a single degree of freedom system is expressed using Newton's second law in Equation 2-29:

$$M\ \ddot{x}(t) + C\ \dot{x}(t) + K\ x(t) = f(t) \tag{2-29}$$

For the general case with a forcing function that can be represented as a summation of sin and cosine terms, the forcing function can be represented as:

$$f(t) = \sum_{\omega=0}^{\infty} F(\omega)\ e^{j\omega t} \tag{2-30}$$

Assuming that the system is underdamped and that enough time has passed that any transient response of the system due to initial condition or startup of the excitation has decayed to zero, the response of the system can be represented as:

$$x(t) = \sum_{\omega=0}^{\infty} X(\omega)\ e^{j\omega t} \tag{2-31}$$

Note that, while $x(t)$ and $f(t)$ are real valued functions, $X(\omega)$ and $F(\omega)$ are complex valued. Working with any arbitrary frequency term in Equations 2-30 and 2-31, Equation 2-30, Equation 2-31 and the derivatives of Equation 2-31 can substituted into Equation 1 yielding the following frequency response function (FRF) relationship for a SDOF system:

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{-M\ \omega^2 + C\ j\omega + K} \tag{2-32}$$

Note the characteristic of the above frequency response function when it is evaluated (measure) at the undamped natural frequency. At the undamped natural frequency, the mass and stiffness terms cancel each other and the FRF is purely imaginary valued.

The first extension that is necessary provides a description for the case where $x(t)$ and $f(t)$ are not located at the same point. On a single degree-of-freedom system, this would provide redundant information (no new information) but it becomes important as the extension to multple degrees-of-freedom occurs. For example, assume that the particular point (and direction) on the mass where the force is applied is referred to as DOF p and the particular point (and direction) on the mass where the response is measured is referred to as DOF q. Equation 2-32 now can be written as follows to note this information.

$$H_{qp}(\omega) = \frac{X_q(\omega)}{F_p(\omega)} = \frac{1}{-M\ \omega^2 + C\ j\omega + K} \tag{2-33}$$

The system is still a SDOF system so $H_{qp} = H_{pp} = H_{qq} = H_{qs} = \cdots$ but the input and output location can now be described. This clearly demonstrates that the number of modes (one in this case) is unrelated to the number of input and output sensors that are used to measure the system.

The second extension that is necessary provides a way to indicate that the modal characteristics (modal coefficients) of both the input and output are represented in the frequency response function model. The modal frequency is already represented by noting that the denominator is related to the characteristic equation. A form of modal scaling is already represented by noting the the mass term in the denominator scales the equation. Modal coefficient information, which is relative not absolute information, can be added by changing the numerator to reflect this.

$$H_{qp}(\omega) = \frac{X_q(\omega)}{F_p(\omega)} = \frac{\psi_q\ \psi_p}{-M\ \omega^2 + C\ j\omega + K} \tag{2-34}$$

Note that, since the system is still a SDOF system, the relative motion at each DOF would be normalized to 1 such that $\psi_p = \psi_q = \psi_s = 1$ which shows that Equation 2-33 and 2-34 still represent the same information. Note as before, if the FRF is evaluated (measured) at the undamped natural frequency), the FRF is once again imaginary valued and is a function of the modal coefficients and damping. Assuming the damping is unknown but constant means that the product of the modal coefficients is proportional to the imaginary part of the FRF.

Finally, the third extension that is necessary provides for the change from SDOF to MDOF. Note that for a linear system, linear superposition can be used in the frequency domain to add the information associated with each mode together to represent the frequency response function of a MDOF system. To desribe this, every term in Equation 2-34 will need a subcript (r) to indicate which mode the information is associated with. The final form of the frequency response function is:

$$H_{qp}(\omega) = \frac{X_q(\omega)}{F_p(\omega)} = \sum_{r=1}^{\infty} \frac{\psi_{qr}\ \psi_{pr}}{-M_r\ \omega^2 + C_r\ j\omega + K_r} \qquad (2\text{-}35)$$

Equation 2-35 is one common representation of the FRF of a MDOF system. Note that the M_r, C_r and K_r terms in the denominator are the modal or generalized mass, damping and stifness parameters, not the physical mass, damping and stiffness parameters. The modal or generalized parameters can be found analytically from the physical mass, damping and stiffness parameters or experimentally using more complicated parameter estimation algorithms.

Note that, as long as the modes are well separated in frequency, the information in the neighborhood of the undamped natural frequency for a given mode can be found from:

$$H_{qp}(\omega) = \frac{X_q(\omega)}{F_p(\omega)} \approx \frac{\psi_{qr}\ \psi_{pr}}{M_r\ \omega^2 + C_r\ j\omega + K_r} \qquad (2\text{-}36)$$

Note that, if the output DOF (point and location) is held fixed while the input DOF is moved, the only information that changes in Equation 2-36 as different FRFs are measured is the information relative to the modal coefficient for the particular mode of interest. If Equation 2-36 is evaluated (measured) near the undamped natural frequency, this means that the imaginary part of the FRF will be proportional to the modal coefficient. The proportionality constant α_r) is:

$$\alpha_r \approx \frac{\psi_{pr}}{-M_r\ \omega^2 + C_r\ j\omega + K_r} \tag{2-37}$$

Since mode shapes are relative patterns, not absolute motions, the value of the constant is not important unless damping or modal scaling is required.

Therefore, modal vectors can be estimated from the imaginary part of the frequency response functions at the damped natural frequencies (or from the magnitude and phase information of the frequency response functions at the damped natural frequencies). This will be reasonably accurate as long as the undamped natural frequencies are well separated and the damping is small (undamped and damped natural frequencies nearly equal).

This result is consistent with the expansion theorem concept (the response of the system at any instant in time or at any frequency is a linear combination of the modal vectors):

Expansion Theorem - Time Domain:

$$\{\ x(t_i)\ \} = \sum_{r=1}^{N} \beta_r\ \{\ \psi_r\ \} \tag{2-38}$$

Expansion Theorem - Frequency Domain:

$$\{\ X(\omega_i)\ \} = \sum_{r=1}^{N} \beta_r\ \{\ \psi_r\ \} \tag{2-39}$$

Using the frequency domain form of the expansion theorem, if the response is evaluated at the undamped natural frequency of mode r, the expansion coefficient β_r will dominate and be approximately equal to alpha defined in Equation 2-37.