8. ADVANCED MODAL ANALYSIS CONCEPTS

8.1 Introduction

As the theoretical basis of expermental modal analysis is extended to real world problems, several clarifications of the theory developed to the present time must be made. The development of more general and/or concise models to represent the entire frequency response function matrix $[H(\omega)]$ or impulse response function matrix $[h(t)]$ is the primary concern. These models must be consistent with the single reference concepts developed previously but must be compatible with multiple reference concepts as well. Therefore, the general concept of measurement degree of freedom must be extended to account for the multiple input, multiple output nature of the problem. Two other concepts must also be discussed in order to fully develop the theoretical basis for experimental modal analysis. These concepts include systems that have repeated modal frequencies (repeated roots) and systems that can not be considered reciprocal.

8.2 Measurement Degrees of Freedom

For the general situation of a multiple input, multiple output model of a system, the experimental definition of the mechanical system is generated from the frequency, or impulse, response function matrix. The size of this matrix is a function of the locations where forces are applied to the mechanical system (inputs) and a function of the locations where responses of the mechanical system (outputs) are measured. This general concept is often referred to as *measurement degrees of freedom* to distinguish the size of the matrix from the number of modal frequencies N of the mechanical system. Obviously, since there is no reason to assume that the number of inputs will be the same as the number of outputs, this general concept of measurement degrees of freedom needs to be extended to properly reflect that the dimension of the frequency, or impulse, response function matrix is rectangular. With this in mind, the number of inputs can be defined by N_i and the number of outputs can be defined by N_o. Therefore, the dimension of the frequency, or impulse, response function matrix is $N_o \times N_i$.

8.3 Mathematical Models

The mathematical model that represents the relationship between the modal parameters and the measured frequency, or impulse, response functions can be represented as follows:

Frequency Response Function Model

Single Measurement:

$$\sum_{k=0}^{m} \alpha_k \, (j\omega)^k H_{pq}(\omega) = \sum_{k=0}^{n} \beta_k \, (j\omega)^k \tag{8.1}$$

$$H_{pq}(\omega) = \sum_{r=1}^{N} \frac{A_{pqr}}{j\omega - \lambda_r} + \frac{A^*_{pqr}}{j\omega - \lambda^*_r} \tag{8.2}$$

Multiple Measurement:

$$\sum_{k=0}^{m} \left[[\alpha_k] \, (j\omega)^k \right] [H(\omega)] = \sum_{k=0}^{n} \left[[\beta_k] \, (j\omega)^k \right] [I] \tag{8.3}$$

$$[H(\omega)\,]_{N_o \times N_i} = \sum_{r=1}^{N} \frac{\left[A_r \right]}{j\omega - \lambda_r} + \frac{\left[A^*_r \right]}{j\omega - \lambda^*_r} \tag{8.4}$$

Impulse Response Function Model

Single Measurement:

$$h_{pq}(t) = \sum_{r=1}^{N} A_{pqr}\; e^{\lambda_r t} + A^*_{pqr}\; e^{\lambda^*_r t} \tag{8.5}$$

Multiple Measurement:

$$[h(t)\,]_{N_o \times N_i} = \sum_{r=1}^{N} \Big[\, A_r \,\Big] e^{\lambda_r t} + \Big[\, A^*_r \,\Big] e^{\lambda^*_r t} \tag{8.6}$$

where:

- $s =$ Laplace variable
- $s = \sigma + j\omega =$ Angular damping variable (rad/sec)
- $\omega =$ Angular frequency variable (rad/sec)
- $p =$ Measured degree-of-freedom (response)
- $q =$ Measured degree-of-freedom (input)
- $r =$ Modal vector number
- $m =$ Number of poles or modal frequencies (2N)
- $n =$ Number of zeroes (2N-2 or less)
- $N =$ Number of positive modal frequencies
- $A_{pqr} =$ Residue $= Q_r \psi_{pr} \psi_{qr}$
- $Q_r =$ Complex modal scaling coefficient for mode r
- $\psi_{pr} =$ Modal coefficient for measured degree-of-freedom p and mode r
- $[A_r] =$ Residue matrix for mode r ($N_o \times N_i$)
- $\lambda_r =$ System pole $= \sigma_r + j\omega_r$

While these models are perfectly appropriate for the multiple input, multiple output case, by a slight alteration of these models a more appropriate form of the models can be developed which will facilitate the development of parameter estimation algorithms.

First of all, the summation form of the equations can be simplified from two terms to one term as follows:

$$[H(\omega)\,] = \sum_{r=1}^{2N} \frac{\left[A_r \right]}{j\omega - \lambda_r} \tag{8.7}$$

$$[h(t)\,] = \sum_{r=1}^{2N} \left[A_r \right] e^{\lambda_r t} \tag{8.8}$$

If these forms of the mathematical models are used, no assumption is made in the model concerning the complex conjugate nature of the solution for modal frequencies (λ_r) or modal vectors ($\{ \psi_r \}$). When the modal parameters are estimated, the evaluation of modal parameters can include a comparison of these terms to determine whether the complex conjugate nature of the solution is found.

Finally, the summation in the mathematical models can be eliminated completely if a different form of the residue is used. In order to do this, the concept of *modal participation factor* (L_{qr}) is introduced. Physically, the modal participation factor is a relative indication of how well a particular mode of vibration is excited from a specific measurement degree of freedom. If all of the modal participation factors for a specific modal vector are represented in a row, this vector is referred to as the modal participation vector and has dimension of $1 \times N_i$. The modal participation vector is not unique (has properties of an eigenvector) but in combination with the modal coefficient defines the residue in the following way.

$$A_{pqr} = Q_r \psi_{pr} \psi_{qr} \tag{8.9}$$

$$A_{pqr} = L_{qr} \psi_{pr} \tag{8.10}$$

$$\{A\,\}_{qr} = L_{qr} \{\psi\,\}_r \tag{8.11}$$

$$L_{qr} = Q_r \psi_{qr} \tag{8.12}$$

Equations 8.10 and 8.11, therefore, are the general statements relating the residue, the modal participation factor, and the modal coefficient. For the general case, the modal coefficient can be thought of as an element from the right eigenvector of the system; the modal participation factor can be thought of as an element from the left eigenvector of the system. With this in mind, when a system obeys Maxwell's Reciprocity, the right and left eigenvectors represent the same modal vector, Equations 8.9 and 8.12 are valid. For the general (nonsymmetric) case, though, only Equations 8.10 and 8.11 will be valid.

From the above equations it is important to note that only the residue can be absolutely and uniquely defined. Both the modal coefficient ψ_{pr} and the modal participation factor L_{qr} are a function of one another and can take on any value; only the combination of the two terms is unique.

Since the term *modal mass* can be defined in terms of the modal scaling Q_r, the modal mass can also be defined in terms of the modal participation factor L_{qr}.

$$M_r = \frac{\psi_{qr}}{2\, j L_{qr}\, \omega_r} \tag{8.13}$$

If this simplification is made, the frequency response function model can now be written in the following form:

$$[H(\omega)]_{N_o \times N_i} = \left[\psi\right]_{N_o \times 2N} [\Lambda]_{2N \times 2N} [L]_{2N \times N_i} \tag{8.14}$$

where:

$$[\Lambda]_{2N \times 2N} = \begin{bmatrix} \frac{1}{j\omega - \lambda_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{j\omega - \lambda_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{j\omega - \lambda_r} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{j\omega - \lambda_{2N}} \end{bmatrix}$$

Likewise, a similar simplification can be developed for the impulse response function matrix:

$$[h(t)]_{N_o \times N_i} = \left[\psi \right]_{N_o \times 2N} \left[e^{\lambda_r t} \right]_{2N \times 2N} \left[L \right]_{2N \times N_i} \tag{8.15}$$

where:

$$\left[e^{\lambda_r t} \right]_{2N \times 2N} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_r t} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\lambda_{2N} t} \end{bmatrix}$$

Since the modal participation vector $\{ L \}_r$ is unique only to within a complex scaling constant, several other definitions of the modal participation factors are possible. One common form can be defined by scaling the modal participation such that a specific element of the vector, L_{pr}, is unity. If this normalization is used, the following definition of modal participation factor will apply:

$$\overline{L_{pqr}} = \frac{L_{qr}}{L_{pr}} = \frac{Q_r \psi_{qr}}{Q_r \psi_{pr}} = \frac{\psi_{qr}}{\psi_{pr}} \tag{8.16}$$

This form of the definition of modal participation factor is used in the development of the theory for the Polyreference Time Domain modal parameter estimation algorithm.

8.4 Repeated Modal Frequencies

Repeated modal frequencies occur whenever two or more modes of the system occur at exactly the same modal frequency λ_r. This condition is often also referred to as repeated roots or repeated poles. Analytically, the presence of repeated roots is determined directly from the characteristic equation just as in any other case. The modal vectors, associated with the repeated roots, now come from a system of equations (the homogeneous equations evaluated at a repeated root) which will be rank deficient by more than one. This means that rather than choosing one of the physical coordinates, as is the case for the non-repeated root situation, a number of physical coordinates, equal to the number of repeated roots, must be assumed. This process is repeated once for each of the repeated roots. Each of the vectors found in this manner will in general be independent to one another and orthogonal to all other modal vectors. Each vector within this set, however, will not necessarily be orthogonal to each other. While this is not a problem mathematically, it may be more consistent to find a set of vectors that are also orthogonal to one another. This can be accomplished by additionally imposing the cross-orthogonality constraints between the vectors. Using any mathematical procedure to determine a set of orthogonal vectors from a set of independent vectors will also work as long as the weighting matrix (mass and/or stiffness) is utilized in the procedure. Note that there is an infinite number of modal vector sets that will satisfy repeated root situation, whether the orthogonality constraint is enforced or not.

Experimentally, it is important to detect the presence of repeated roots in order to build a complete modal model of the mechanical system that will accurately represent the dynamic response of the mechanical system to any set of forcing conditions. The most common cause of this situation is symmetry in the mechanical system. For example, in the case of a flagpole, there are two modes of vibration occurring at the same frequency for each lateral bending mode of the flagpole. Any time one or more axes of symmetry exist in the mechanical system, this condition will exist. The important consideration, though, is that, in order to detect the repeated root condition, more than one row or column of the frequency, or impulse, response function matrix must be measured. Therefore, to detect a repeated root of order two, two *independent* rows or columns of the frequency, or impulse, response function matrix must be used. Note that it is possible to choose two rows or columns that are not independent and thus miss the repeated root.

The modal vectors that are associated with repeated modal frequencies are independent of one another and each is orthogonal (weighted) to the other modal vectors of the set. Even so, the modal vectors associated with the repeated modal frequencies do not individually have fixed patterns even in the relative sense that modal vectors associated with nonrepeated modal frequencies do. Only when the set of modal vectors associated with repeated modal frequencies are considered as a set, are the modal vectors unique in any way. In the case of a repeated modal frequency of multiplicity two, two independent modal vectors will be required to describe the modal space but any two independent modal vectors will do. An analogy in three dimensional graphics involves using two vectors to define a plane. There is an infinite set of vectors that can be used to describe the same plane. Any two vectors lying in that plane can be used to uniquely define the plane and yet the two vectors are not unique but only independent from one another.

In this case, independent simply means that the two vectors are not scalar multiples of one another.

Therefore, there are an infinite number of combinations of modal vectors which will serve as the modal vectors for the repeated modal frequencies. In this case, the exact characteristic of each modal vector is unimportant since the set of modal vectors will always be considered together. The modal vectors associated with the repeated modal frequencies often appear to be exactly the same at first glance. This is due to the symmetry and on close inspection using physical coordinates and directions, the distinction can be easily detected. Again consider the flagpole example. Since there is no way to define a unique x and y direction with respect to the circular cross section of the flagpole, there will not be a single set of modal vectors that will describe the modal deformation at a repeated modal frequency.

At this point, repeated modal frequencies may seem to be only a theoretical concept that does not have much impact on real structures that are not symmetric. Actually, due to the discrete nature of the frequency response function, a very real problem often exists where several modal frequencies occur between the frequency resolution that is used. Since *a priori* knowledge of the modal density is not normally possible, this condition happens quite frequently. This is referred to as *psuedo-repeated modal frequencies.* While the modal vectors do not theoretically have the same attributes as in the repeated root case, for all practical purposes, the result is the same.

For the case of a mechanical system with repeated roots, or psuedo-repeated roots, Leuridan [1] has shown that the same mathematical model can be used to represent the relationship between measured frequency, or impulse, response function data and modal parameters. For a mechanical system with repeated modal frequencies of order N_r the following basic relationship for frequency response functions will apply.

$$H_{pq}(\omega) = \sum_{r=1}^{2N} \frac{A_{pqr}}{j\omega - \lambda_r} \tag{8.17}$$

$$H_{pq}(\omega) = \frac{A_{pq1}}{j\omega - \lambda_1} + \frac{A_{pq2}}{j\omega - \lambda_2} + \,.\,.\,.\,.\,.\,.\,.\,. + \sum_{s=1}^{N_r} \frac{A_{pqs}}{j\omega - \lambda_s} + \,.\,.\,.\,.\,.\,.\,.\,. + \frac{A_{pqr}}{j\omega - \lambda_r} + \,.\,.\,.\,.\,.\,.\,.\,. \tag{8.18}$$

With the above model, several poles could be repeated; the multiplicity of any pole can be at most N.

In order to understand the implications of the repeated modal frequency on the residue information, the above equation can be rewritten in terms of column q of the frequency response function matrix.

$$\{H\}_q = \frac{Q_1 \psi_{q1} \{\psi\}_1}{j\omega - \lambda_1} + \frac{Q_2 \psi_{q2} \{\psi\}_2}{j\omega - \lambda_2} + \ldots\ldots + \sum_{s=1}^{N_r} \frac{Q_s \psi_{qs} \{\psi\}_s}{j\omega - \lambda_s} + \ldots\ldots + \frac{Q_r \psi_{qr} \{\psi\}_r}{j\omega - \lambda_r} + \ldots\ldots \tag{8.19}$$

The implication of this representation is that if only one column of the frequency response function matrix is measured, the residue that will be estimated for the repeated modal frequencies will be the linear combination represented by the summation in the above equation [1]. Note that this linear combination is not unique; if a different column of the frequency response function is used, the residue column for the repeated modal frequencies will not be the same. This observation yields the simplest procedure for the detection of repeated or psuedo-repeated modal frequencies. If the modal vectors that are estimated from different reference positions (different inputs) are not the same, the modal vectors are probably the result of repeated modal frequencies. Note that the modal vectors associated with the repeated modal frequencies are independent of one another and not simply a scalar multiple of one another as would be the case for a nonrepeated modal frequency.

Therefore, if the repeated modal frequency condition is not detected, the residue vector, associated with the repeated modal frequency, that will be estimated from the frequency response function data taken from only one reference will represent a linear combination of the modal vectors associated with the repeated modal frequencies. If frequency response function data from a second reference is observed without knowledge of the first reference (and the repeated modal frequency condition is still not detected) the residue vector will again represent a linear combination of the modal vectors associated with the repeated modal frequencies. Unfortunately, this residue vector will not represent the same modal vector when compared to the previous estimate. This means that knowledge of any single reference set of data will be insufficient to describe the repeated root situation.

Nevertheless, if data from several references is available, a characteristic of the modal vectors associated with the repeated root is possible. In general, for a repeated root of order N_r, at least N_r references will be required to make this determination. Unlike in the non-repeated condition, though, the individual modal vectors determined by this process will not be unique. Only the combination of vectors will represent a unique characteristic. For example, if a system contains a repeated modal frequency λ_s of order 2, the two residue columns that will be estimated for columns p and q can always be represented as follows:

For column p:

$$\{A\}_p = Q_{11}\psi_{p1}\{\psi\}_1 + Q_{22}\psi_{p2}\{\psi\}_2 \tag{8.20}$$

For column q:

$$\{A\}_q = Q_{11}\psi_{q1}\{\psi\}_1 + Q_{22}\psi_{q2}\{\psi\}_2 \tag{8.21}$$

Therefore, in matrix notation:

$$\left[\{A\}_p\{A\}_q\right] = \left[\{\psi\}_1\{\psi\}_2\right]\begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22}\end{bmatrix}\begin{bmatrix}\psi_{p1} & \psi_{q1} \\ \psi_{p2} & \psi_{q2}\end{bmatrix} \tag{8.22}$$

The above equation states that, when the modal vectors, $\{\psi\}_1$ and $\{\psi\}_2$, are excited in *independent* combinations in column p and column q of the frequency response function matrix, then the residue vectors, $\{A\}_p$ and $\{A\}_q$, are independent and in turn define two independent modal vectors that always can be normalized such that the modal scaling matrix, $[Q]$, is diagonal.

For the general case of a repeated modal frequency of order N_r, there will be a set of N_r

independent modal vectors that will satisfy Eq. 8.22. Note that any set of N_r modal vectors that are independent linear combinations of the modal vectors defined by Eq. 8.22 can be used. In this case, the modal scaling matrix, [Q], will not be diagonal. This case is represented by the following equation and will be demonstrated in a later example.

$$\left[\{A\}_p \{A\}_q \right] = \left[\{\psi\}_1 \{\psi\}_2 \right] [Q] \begin{bmatrix} \psi_{p1} & \psi_{q1} \\ \psi_{p2} & \psi_{q2} \end{bmatrix} \tag{8.21}$$

8.4.1 Repeated Modal Frequency Example: Residue Synthesis

In order to fully understand the nature of the problem that can arise due to repeated modal frequencies, a simple example may be used. Consider a repeated modal frequency, λ_s, of order two in a three degree of freedom system. The following represents two independent modal vectors associated with the two repeated modal frequencies.

$$\{\psi\}_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

$$\{\psi\}_2 = \begin{Bmatrix} 2 \\ 1 \\ 3 \end{Bmatrix}$$

For this example, assume that the modal vectors given above have been scaled such that the modal scaling matrix, [Q], is diagonal with elements on the diagonal equal to j.

$$[Q] = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}$$

For the following discussion, only the portion of the frequency response function matrix, [H], that depends upon the repeated modal frequencies will be synthesized.

To begin with, all of the columns of the frequency response function matrix will be synthesized from the theoretical modal data (Case 1, 2, 3). This data represents the true answer or the answer that would be generated from the measured frequency response functions if the repeated root condition was not detected.

Case Number One: Column 1 of $[H\ ]$

Case One represents the proper synthesis of the repeated root portion of Column 1 of the frequency response function matrix.

$$\{H\ \}_1 = \ldots\ldots\ldots + \frac{j1 \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}_1}{j\omega - \lambda_s} + \frac{j2 \begin{Bmatrix} 2 \\ 1 \\ 3 \end{Bmatrix}_2}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\ \}_1 = \ldots\ldots\ldots + \frac{j \begin{Bmatrix} 5 \\ 4 \\ 9 \end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\ \}_1 = \ldots\ldots\ldots + \frac{j\ \sqrt{5} \begin{Bmatrix} \sqrt{5} \\ \frac{4}{\sqrt{5}} \\ \frac{9}{\sqrt{5}} \end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Notice that if only this column of the frequency response function matrix is measured, there is no reason to suspect that a repeated modal frequency exists. Therefore, the number of modal frequencies and modal vectors would be reduced accordingly.

Case Number Two: Column 2 of $[H]$

Case Two represents the proper synthesis of the repeated root portion of Column 2 of the frequency response function matrix.

$$\{H\}_2 = \ldots\ldots\ldots + \frac{j2\begin{Bmatrix}1\\2\\3\end{Bmatrix}_1}{j\omega-\lambda_s} + \frac{j1\begin{Bmatrix}2\\1\\3\end{Bmatrix}_2}{j\omega-\lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_2 = \ldots\ldots\ldots + \frac{j\begin{Bmatrix}4\\5\\9\end{Bmatrix}}{j\omega-\lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_2 = \ldots\ldots\ldots + \frac{j\sqrt{5}\begin{Bmatrix}\frac{4}{\sqrt{5}}\\\sqrt{5}\\\frac{9}{\sqrt{5}}\end{Bmatrix}}{j\omega-\lambda_s} + \ldots\ldots\ldots$$

Once again notice that if only this column of the frequency response function matrix is measured, there is no reason to suspect that there is a repeated modal frequency. If the modal vector for the modal frequency λ_s is compared to that found from column 1 of the frequency response function matrix, it is obvious that a different modal vector has been estimated. This comparison is the simplest method of detecting a repeated modal frequency.

Case Number Three: Column 3 of $[H\ ]$

Case Three represents the proper synthesis of the repeated root portion of Column 3 of the frequency response function matrix.

$$\{H\ \}_3 = \ldots\ldots\ldots + \frac{j3\begin{Bmatrix}1\\2\\3\end{Bmatrix}_1}{j\omega-\lambda_s} + \frac{j3\begin{Bmatrix}2\\1\\3\end{Bmatrix}_2}{j\omega-\lambda_s} + \ldots\ldots\ldots$$

$$\{H\ \}_3 = \ldots\ldots\ldots + \frac{j\begin{Bmatrix}9\\9\\18\end{Bmatrix}}{j\omega-\lambda_s} + \ldots\ldots\ldots$$

$$\{H\ \}_3 = \ldots\ldots\ldots + \frac{j\ \sqrt{18}\begin{Bmatrix}\frac{9}{\sqrt{18}}\\\frac{9}{\sqrt{18}}\\\sqrt{18}\end{Bmatrix}}{j\omega-\lambda_s} + \ldots\ldots\ldots$$

Notice that the modal vector determined from this column of the frequency response function is once again different from either of the first two columns. This is a characteristic of the modal vector resulting from a repeated modal frequency; if the repeated modal frequency is not detected, the modal vector appears to be different as a function of the reference (input) location.

If only the first column of the frequency response function matrix $[H\ ]$ has been measured and the repeated modal frequency has not been identified, the synthesis of column 2 and column 3 can be attempted from the measured first column as follows:

Case Number Four: Column 2 of $[H]$

Case Four represents the improper synthesis Column 2 of the frequency response function if only frequency response function data from Column 1 is used (repeated root not detected).

$$\{H\}_2 = \ldots\ldots\ldots + \frac{j\,\frac{4}{\sqrt{5}}\begin{Bmatrix}\sqrt{5}\\ \frac{4}{\sqrt{5}}\\ \frac{9}{\sqrt{5}}\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_2 = \ldots\ldots\ldots + \frac{j\begin{Bmatrix}4\\ \frac{16}{5}\\ \frac{36}{5}\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Note that this synthesis does not agree with Case 2 which is the theoretical result. Therefore, if only column 1 of the frequency response function matrix is measured, the repeated modal frequency cannot be detected and the proper dynamic characteristics of the system, when excited at location 2, cannot be predicted.

Case Number Five: Column 3 of $[H\ ]$

Case Five represents the improper synthesis Column 3 of the frequency response function if only frequency response function data from Column 1 is used (repeated root not detected).

$$\{H\ \}_3 = \ldots\ldots\ldots + \frac{j\,\dfrac{9}{\sqrt{5}}\begin{Bmatrix}\sqrt{5}\\[4pt] \dfrac{4}{\sqrt{5}}\\[4pt] \dfrac{9}{\sqrt{5}}\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\ \}_3 = \ldots\ldots\ldots + \frac{j\begin{Bmatrix}9\\[4pt] \dfrac{36}{5}\\[4pt] \dfrac{81}{5}\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Once again the result for this case should compare to Case 3 (the theoretical case) but does not. Therefore, if only column 1 of the frequency response function matrix is measured, the repeated modal frequency cannot be detected and the proper dynamic characteristics of the system, when excited at location 3, cannot be predicted.

8.4.2 Repeated Modal Frequency Example: Modal Vector Solution

Using the residue data generated from two independent columns of the frequency response function matrix, the set of independent modal vectors for the repeated modal frequency can be determined if the order of the repeated modal frequency is known. This can be estimated via singular value decomposition techniques. For the previous example, assuming that the multiplicity of the repeated modal frequency is 2, the results can be determined by using the first two columns of the frequency response function matrix (Case 1 and Case 2).

$$\left[\{H\}_1 \; \{H\}_2 \right] = \ldots\ldots\ldots + \frac{j \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 9 & 9 \end{bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Since the residue columns found from the two columns are independent, these residue columns can be used as the [ψ] matrix in Eq. 8.21. In this case, though, there is no reason to assume that the modal scaling matrix [Q] is diagonal. With this in mind, Eq. 8.21 can be used in Eq. 8.2 as follows:

$$\left[\{H\}_1 \; \{H\}_2 \right] = \ldots\ldots\ldots + \frac{j \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 9 & 9 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

While it may not be immediately obvious, the modal scaling matrix in the above equation must be chosen so that the product of the three matrices and the scalar j yields the proper residues for the first two columns. The inverse of this matrix is the only matrix that satisfies this condition.

$$\left[\{H\}_1 \; \{H\}_2 \right] = \ldots\ldots\ldots + \frac{j \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 9 & 9 \end{bmatrix} \begin{bmatrix} 5/9 & -4/9 \\ -4/9 & 5/9 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Now using these two residue columns and the modal scaling matrix [Q] the synthesis of the residue for column 3 is possible.

$$\{H\}_3 = \ldots\ldots\ldots + \frac{j \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 9 & 9 \end{bmatrix} \begin{bmatrix} 5/9 & -4/9 \\ -4/9 & 5/9 \end{bmatrix} \begin{bmatrix} 9 \\ 9 \end{bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_3 = \ldots\ldots\ldots + \frac{j \left[\begin{matrix} 9 \\ 9 \\ 18 \end{matrix} \right\}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

This is the correct answer as previously defined by Case 3. Note that it is possible to find a set of modal vectors such that the corresponding modal scaling matrix [Q] is diagonal. This is done by substituting the eigenvalue decomposition of [Q] into the above equation. While this is possible and makes the theory consistent, it is unnecessary as the previous example proved.

Question?

If the modal vectors corresponding to a repeated modal frequency with multiplicity of two are:

$$\{\psi\}_1 = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$$

$$\{\psi\}_2 = \begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix}$$

Is it possible to correctly identify the modal vectors from the first and third columns of the frequency response function matrix if the multiplicity of the repeated modal frequency is known? (Hint: Are the first and third columns of the frequency response function matrix independent?)

8.5 Left and Right Eigenvectors

The relationship between the modal participation vector and the modal vector is particularly interesting when the theoretical matrix equation of motion is examined. For the case of a system that obeys Maxwell's reciprocity, the mass, stiffness, and damping matrices will be symmetric matrices. With this in mind, the basic eigenvalue-eigenvector problem that is generated is of the following form:

$$\left[\lambda_r^2 [M] + \lambda_r [C] + [K] \right] \{\psi\}_r = \{0\} \tag{8.22}$$

In the above equation, note that the dimension of $\{\psi\}_r$ is the same as the dimension of the square mass, stiffness, or damping matrix. Since this theoretical problem is of dimension $N \times N$, there is no difference between $N,\ N_{o,}\ or\ N_i$. Therefore, the modal participation vector is also of the same dimension.

The eigenvector $\{\ \psi\ \}_r$ is also referred to as the right eigenvector of the system. If the eigenvalue-eigenvector problem is reformulated, the eigenvector is also the left eigenvector of the system.

$$\{\psi\ \}_r^T \left[\ \lambda_r^2[M\ ] + \lambda_r[C\ ]\ + [K\ ]\ \right] = \{0\ \} \tag{8.23}$$

Note that in both Eqs. 8.22 and 8.23, the modal vector $\{\ \psi\ \}_r$ could be replaced by the modal participation vector $\{\ L\ \}_r$ without loss of generality.

If the system does not satisfy reciprocity (the mass, stiffness, and/or damping matrices are not symmetric), then the right and left eigenvectors will be different. For the right eigenvector, Eq. 8.24 will be appropriate.

$$\left[\ \lambda_r^2[M\ ] + \lambda_r[C\ ]\ + [K\ ]\ \right]\{\psi\ \}_r = \{0\ \} \tag{8.24}$$

For the left eigenvector, Eq. 8.25 will now be the appropriate form.

$$\{L\ \}_r^T \left[\ \lambda_r^2[M\ ] + \lambda_r[C\ ]\ + [K\ ]\ \right] = \{0\ \} \tag{8.25}$$

Note that the modal participation vector is the same as the left eigenvector. At this point it is important to note that the residue for a nonreciprocal system can still be defined by the product of the modal vector and the modal participation factor as in Eqs. 8.8 and 8.9. For the nonreciprocal case, the relationship between the modal vector and modal participation vector, as defined by Eq. 8.10, no longer is true.

This concept of the relationship between left and right eigenvectors and modal vectors and modal participation vectors is true for systems that have repeated modal frequencies as well.

8.6 Summary/Conclusions

Whether a system contains repeated modal frequencies or nonreciprocal characteristics, the general equations relating modal parameters to the measure frequency, or impulse, response function matrices will properly predict the dynamics of the system as long as sufficient elements of the matrix are measured. The extension of Eqs. 8.1 and 8.3 to Eqs. 8.12 and 8.13 provide the generality and flexibility to describe these characteristics.

8.7 References

[1] Leuridan, J., *Some Direct Parameter Model Identification Methods Applicable for Multiple Input Modal Analysis*, Doctoral Dissertation, Department of Mechanical Engineering, University of Cincinnati, 1984, 384 pp.